Math 51 Midterm 1 Solutions (Feb, 2010)

1. Complete the following definitions.

(a). A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbf{R}^n$ is called *linearly dependent* provided

> one of the vectors can be written as a linear combination of the other vectors.

or:

> there are scalars $c_1, c_2, \ldots, c_k$, not all 0, such that
> $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0.$$

(b). A set V of vectors in $\mathbf{R}^n$ is called a *linear subspace* provided

> it contains $\mathbf{0}$, it is closed under addition, and it is closed under scalar multiplication.

(c). A map $T : \mathbf{R}^n \to \mathbf{R}^k$ is called a *linear* map provided

> $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, and
> $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $c \in \mathbf{R}$ and $\mathbf{x} \in \mathbf{R}^n$.

or

> T commutes with addition and with scalar multiplication.

(d). A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors in a linear subspace V is called a *basis* for V provided

> the vectors span V and they are linearly independent.

(e). The *dimension* of a subspace V is

> the number of vectors in a basis for V.

2. Find the row reduced echelon form rref(A) of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 5 \\ 2 & 4 & 7 & 10 & 8 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 2 & 4 & 7 & 10 & 8 \\ 0 & 2 & 0 & 2 & 5 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 7/2 & 5 & 4 \\ 0 & 2 & 0 & 2 & 5 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7/2 & 3 & -1 \\ 0 & 2 & 0 & 2 & 5 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 7/2 & 3 & -1 \\ 0 & 1 & 0 & 1 & 5/2 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7/2 & 3 & -1 \\ 0 & 1 & 0 & 1 & 5/2 \\ 0 & 0 & 0 & 0 & -1/2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 7/2 & 3 & -1 \\ 0 & 1 & 0 & 1 & 5/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7/2 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3(a). Consider the following matrix B and its row reduced echelon form $\mathrm{rref}(B)$:

$$B = \begin{bmatrix} 4 & 3 & 7 & 0 & 3 \\ 2 & 3 & 5 & 0 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 5 & 4 & 9 & 0 & 4 \end{bmatrix} \quad , \quad \mathrm{rref}(B) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(You do not need to check this.) Find a basis for the column space $C(B)$ of B.

The pivots in $\mathrm{rref}(B)$ are in columns 1, 2, and 5, so the corresponding columns of B form a basis:

$$\begin{bmatrix} 4 \\ 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}.$$

3(b). Find a basis for the nullspace $N(B)$ of B (where B is as in part (a)).

Solution: From $\mathrm{rref}(B)$, we see that $\mathbf{x} \in N(B)$ if and only

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \\ x_5 &= 0 \end{aligned}$$

or (moving free variables to the right):

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \\ x_5 &= 0 \end{aligned}$$

or (in vector form):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4$$

so $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ form a basis for $N(B)$.

4(a). Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 1 & 2 \\ 3 & 10 \end{bmatrix}$. Find the condition(s) on a vector **b** for **b** to be in the column space of A. (Your answer should be one or more equations involving the components b_i of **b**.)

Solution: We do the row reduced echelon form for the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 2 & 7 & b_2 \\ 1 & 2 & b_3 \\ 3 & 10 & b_4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & -1 & b_3 - b_1 \\ 0 & 1 & b_4 - 3b_1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 7b_1 - 3b_2 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_2 - 3b_1 \\ 0 & 0 & b_4 - b_2 - b_1 \end{array}\right]$$

Therefore the conditions for vector **b** are

$$\boxed{\begin{array}{l} b_3 + b_2 - 3b_1 = 0 \quad \text{and} \\ b_4 - b_2 - b_1 = 0. \end{array}}$$

4(b). Find a matrix B such that $N(B) = C(A)$. (Here A is the matrix in part (a).)

Solution: We can rewrite the conditions from part (a) as

$$\begin{aligned} -3b_1 + b_2 + b_3 + 0b_4 &= 0 \\ -b_1 - b_2 + 0b_3 + b_4 &= 0 \end{aligned}$$

or (equivalently) as

$$\begin{bmatrix} -3 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \mathbf{b} = \mathbf{0}.$$

Thus we can let

$$\boxed{B = \begin{bmatrix} -3 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}.}$$

5. Let V be the set of all vectors $\mathbf{x}$ in $\mathbf{R}^4$ that are orthogonal to $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and to $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. (To be in V, a vector must be orthogonal both to $\mathbf{u}$ and to $\mathbf{v}$.) Find a basis for V.

Solution: Let $\mathbf{x}$ be a vector in V. Then

$$\begin{aligned} \mathbf{x} \cdot \mathbf{u} = 0 &\iff x_1 + x_2 + x_3 + x_4 = 0, \quad \text{and} \\ \mathbf{x} \cdot \mathbf{v} = 0 &\iff 2x_1 + 2x_2 + 3x_3 + 4x_4 = 0. \end{aligned}$$

Therefore V is the null space of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \end{bmatrix}.$$

We find the row reduced echelon form of the matrix above.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Therefore $\mathbf{x} \in V$ if and only if

$$\begin{aligned} x_1 + x_2 - x_4 &= 0 \\ x_3 + 2x_4 &= 0, \end{aligned}$$

Thus (moving free variables to the right side and putting in vector form) we that $\mathbf{x} \in V$ if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 + x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} x_4.$$

so we have the basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$

6(a). Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^n$ such that $\|\mathbf{u}\| = \|\mathbf{v}\|$. Prove that the vectors $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ are orthogonal to each other.

Solution: We need to show that $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = 0$.

$$\begin{aligned}(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 \\ &= 0,\end{aligned}$$

where the last equation follows from the fact that $\|\mathbf{u}\| = \|\mathbf{v}\|$. □

6(b). Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent vectors in $\mathbf{R}^n$. Suppose that A is an $m \times n$ matrix. Prove that the vectors $A\mathbf{v}_1$, $A\mathbf{v}_2$, and $A\mathbf{v}_3$ must also be linearly dependent.

Solution 1: Since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent vectors in $\mathbf{R}^n$, there exist c_1, c_2, and c_3, not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Multiplying both sides by A gives

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = \mathbf{0}$$

and therefore

$$(*) \qquad c_1A(\mathbf{v}_1) + c_2A_2(\mathbf{v}_2) + c_3A(\mathbf{v}_3) = \mathbf{0}.$$

Since c_1, c_2, and c_3 are not all 0, equation (*) implies that $A\mathbf{v}_1$, $A\mathbf{v}_2$, and $A\mathbf{v}_3$ are linearly dependent. □

Solution 2: Since the vectors are linearly dependent, one of them, say $\mathbf{v}_3$, a linearly combination of the other two:

$$\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2.$$

Multiplying both sides by A gives $A\mathbf{v}_3 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)$ or

$$A\mathbf{v}_3 = c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2).$$

Thus $A\mathbf{v}_3$ is a linear combination of $A\mathbf{v}_1$ and $A\mathbf{v}_2$, so the vectors $A\mathbf{v}_1$, $A\mathbf{v}_2$, and $A\mathbf{v}_3$ are linearly dependent. □

7(a). Find a parametric equation for the line L through the points $A = (0, 1, 1)$ and $B = (1, 2, 3)$.

Solution: Let the intial point be A. The direction vector is $\overrightarrow{AB} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}$. Therefore we have the parametric representation

$$\boxed{\left\{\begin{bmatrix}0\\1\\1\end{bmatrix} + t\begin{bmatrix}1\\1\\2\end{bmatrix} : t \in \mathbf{R}\right\}.}$$

7(b). Find a point C on L such that the triangle ΔOAC has a right angle at C. (Here $O = (0,0,0)$ is the origin.)

Solution: Since C is a point on L, vector $\overrightarrow{OC}$ is $\begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\1\\2\end{bmatrix}$ for some $c \in \mathbf{R}$. We need to find c. We want to choose c so that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$, i.e., so that

$$0 = \overrightarrow{AB} \cdot \overrightarrow{OC} = \begin{bmatrix}1\\1\\2\end{bmatrix} \cdot \left(\begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\1\\2\end{bmatrix}\right) = 3 + 6c.$$

Thus $c = -1/2$ and hence the point C is

$$\begin{bmatrix}0\\1\\1\end{bmatrix} - 1/2\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\0\end{bmatrix}.$$

8. Suppose $T : \mathbf{R}^3 \to \mathbf{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}7\\13\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\1\\2\end{bmatrix}\right) = \begin{bmatrix}7\\20\end{bmatrix}.$$

Find the matrix for T.

Solution: The matrix for T is

$$\left[T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right].$$

We know that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}7\\13\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}4\\12\end{bmatrix}.$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \frac{1}{2}\left(T\left(\begin{bmatrix}1\\1\\2\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)\right) = \frac{1}{2}\left(\begin{bmatrix}7\\20\end{bmatrix} - \begin{bmatrix}7\\13\end{bmatrix}\right) = \begin{bmatrix}0\\7/2\end{bmatrix}.$$

Therefore the matrix for T is

$$\begin{bmatrix}3 & 4 & 0\\1 & 12 & 7/2\end{bmatrix}.$$

9. Consider the points $A = (1,1,1,1)$, $B = (1,2,0,1)$ and $C = (1,0,1,1)$ in $\mathbf{R}^4$.

9(a). Find the cosine of the angle at B of the triangle ABC.

Solution: To find the cosine of the angle θ at B, we need vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$:

$$\overrightarrow{BA} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\2\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\\0\end{bmatrix} \quad , \quad \overrightarrow{BC} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix} - \begin{bmatrix}1\\2\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-2\\1\\0\end{bmatrix}.$$

From the dot product formula,

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \|\overrightarrow{BA}\|\|\overrightarrow{BC}\| \cos\theta$$
$$3 = \sqrt{2}\sqrt{5}\cos\theta$$

so

$$\boxed{\cos\theta = \frac{3}{\sqrt{10}}} \quad \text{or} \quad \boxed{\cos\theta = \frac{3\sqrt{10}}{10}}.$$

9(b). Find a parametric equation for the plane through the points A, B, and C.

Solution: Let B be an initial point. From (a), we have that the parametric representation

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} : s, t \in \mathbf{R} \right\}.$$

10. Short answer questions. (No explanations required.)

(a). Suppose that a linear subspace V is spanned by vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. What, if anything, can you conclude about the dimension of V?

$$\boxed{\dim(V) \leq k.}$$

(Note: we cannot conclude that $\dim(V) = k$ because the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are not necessarily linearly independent.)

(b). Suppose that a linear subspace W contains a set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ of k linearly independent vectors. What, if anything, can you conclude about the dimension of W?

$$\boxed{\dim(W) \geq k.}$$

(Note: we cannot conclude that $\dim(W) = k$ because the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ do not necessarily span W.)

(c). Suppose $\mathbf{u} \cdot \mathbf{v} < 0$. What, if anything, can you conclude about the angle θ between $\mathbf{u}$ and $\mathbf{v}$? [Note: by definition, the angle θ between two nonzero vectors is in the interval $0 \leq \theta \leq \pi$.]

$$\boxed{\theta > \pi/2.}$$

(This is because $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$, so from $\mathbf{u} \cdot \mathbf{v} < 0$ we conclude that $\cos\theta < 0$.)

(d). Suppose $T : \mathbf{R}^k \to \mathbf{R}^n$ is linear map with matrix A and suppose $\mathbf{b} \in \mathbf{R}^n$. If $k < n$, what, if anything, can you conclude about the number of solutions of $A\mathbf{x} = \mathbf{b}$?

$$\boxed{\text{Nothing.}}$$

Of course because it's a linear system, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, no solutions, or exactly one solution. But $k < n$ gives no *additional* information about the number of solutions (as seen in the following examples), hence the answer "nothing".

- It can have infinitely many solution e.g $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
- It can have no solution e.g $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.
- It can have a unique solution e.g $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Note: "the system has 0 solutions, 1 solution, or infinitely many solutions" is also considered a correct answer.

(e). Suppose V is a 3 dimensional linear subspace of $\mathbf{R}^6$ and suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent vectors in V. What more, if anything, must you know in order to conclude that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for V?

Nothing (that is, you don't need to know anything more.)

(For a set of vectors in a subspace V to be basis for V, the vectors must satisfy two conditions: they must be independent, and they must span V. However, if the number of vectors is equal to the dimension of the subspace, then either condition alone suffices. See proposition 12.3 in the text.)