MATH 51 FINAL EXAM (MARCH 15, 2010)

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Your name (print):

Sign to indicate that you accept the honor code:

Instructions: Find your TA's name in the table above and circle the time that your TTh section meets. During the test, you may not use notes, books, or calculators. Read each question carefully, show all your work, and circle your final answer. Each of the 16 problems is worth 10 points. You have 3 hours to do all the problems. Good luck!

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1. Suppose the temperature at point (x, y) is $f(x, y) = x^2 - 4x + y^2 + 9$.

1(a). Find the hottest point(s) and the coldest point(s) on the ellipse
$$4x^2 + 9y^2 = 36.$$

Solution: We solve this using Lagrange multipliers. $\nabla f = [2x - 4, 2y]$, and $\nabla g = [8x, 18y]$, where g is the function defining the ellipse. From $\nabla f = \lambda \nabla g$, start with the equation $2y = \lambda 18y$, factoring into $y(1 - 9\lambda) = 0$, which breaks us into two cases:
Case $\lambda = \frac{1}{9}$:
The equation $2x - 4 = \lambda 8x$ becomes $x = \frac{18}{5}$. However, every point on the ellipse satisfies $-3 \leqslant x \leqslant 3$, so we conclude this case is impossible.
Case $y = 0$:
Since we are on the ellipse, this tells us $x = \pm 3$. We now plus in $f(-3, 0) = 30$, and $f(3, 0) = 6$, which are the hottest and coldest points.

1(b). Find the hottest point(s) and the coldest point(s) on the region
$$4x^2 + 9y^2 \leq 36.$$

To find points in a region, we need to look for interior critical points, and critical points on the boundary. The boundary calculation is just part (a) though, so we only need to look where $\nabla f = 0$. This only happens at the point $(2, 0)$, and plugging in gives us $f(2, 0) = 5$. So the coldest point in the region is $(2, 0)$, and the hottest point is still $(-3, 0)$.

The most common mistake on part (a) is dividing the equation by y instead of factoring it out, this causes you to miss all solutions where $y = 0$. Also, many students kept $x = \frac{18}{5}$, and either mistakingly got a real answer for y, or wrote down a square root of a negative number. It is also possible to solve part (a) with the parametrization $x = 3\cos\theta$, $y = 2\sin\theta$. (b) was pretty straightforward as long as it was set up right.

2(a). Let S be the surface defined by $x^2y + x^3 + y^2z = 2$. Find an equation for the tangent plane at $(-2, 1, 6)$.

Solution: Notice that S is a level set of the function $f(x, y, z) = x^3 + x^2y + y^2z$. The normal to the tangent plane of any level set is the gradient of the function, thus we calculate $\nabla f = [3x^2+2xy, x^2+2yz, y^2]$ We plug in the point $(-2, 1, 6)$ to get the normal at that point, which gives $[8, 16, 1]$. From here we just need to write the equation of the plane passing through $(-2, 1, 6)$ with normal $[8, 16, 1]$. This plane has equation $8(x + 2) + 16(y - 1) + (z - 6) = 0$, or $8x + 16y + z = 6$.

2(b). Find all points (a, b, c) on S such that the tangent plane to S at (a, b, c) is parallel to the plane $x + y + z = 0$.

Solution: Two planes are parallel if their normal vectors are parallel. So, we just need to find where ∇f and $[1, 1, 1]$ are parallel. This happens if $\nabla f = \lambda[1, 1, 1]$, which is the same as solving $3x^2+2xy = x^2+2yz = y^2$.
First look at $3x^2 + 2xy = y^2$, this factors into $(y + x)(y - 3x) = 0$.
Case $y = -x$:
The equation $x^2 + 2yz = y^2$ tells us that $x^2 - 2xz = x^2$, that is, $xz = 0$. But the equation $x^3 + x^2y + y^2z = 2$ becomes $x^2z = 2$. So, this case is impossible.
Case $y = 3x$
The equation $x^2 + 2yz = y^2$ tells us $x^2 + 6xz = 9x^2$, which says $3xz = 4x^2$. Again, $x = 0$ is impossible, so this tells us that $z = \frac{4}{3}x$. Now, plugging everything into $x^3+x^2y+y^2z = 2$ becomes $x^3+3x^3+12x^3 = 2$, that is, $x = \frac{1}{2}$. This gives us the solution $(x, y, z) = (\frac{1}{2}, \frac{3}{2}, \frac{2}{3})$.

Part (a) had few errors. Part (b) had lots of mistakes, with many people not knowing how to set up the problem (a lot of people started with $\nabla f = [1, 1, 1]$ for example). Most students who set the problem up correctly were stumped by the algebra.
Also, note that it is possible to take the equation defining S, solve for z ($y = 0$ isn't a solution), and work that way.

3. Consider the following system of equations:

$$\begin{aligned} x + 3y + 7z &= b_1 \\ 3x - y + 11z &= b_2 \\ x - y + az &= b_3. \end{aligned}$$

For which values of a does the system have exactly one solution? [Here b_1, b_2, and b_3 are given constants.]

Solution:

$$\det \begin{bmatrix} 1 & 3 & 7 \\ 3 & -1 & 11 \\ 1 & -1 & a \end{bmatrix} = \det \begin{bmatrix} 1 & 3 & 7 \\ 0 & -10 & -10 \\ 0 & -4 & a-7 \end{bmatrix} = -10 \det \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & -4 & a-7 \end{bmatrix}$$

$$= -10 \det \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & a-3 \end{bmatrix} = -10(a-3)$$

From this, the system has exactly one solution when the determinant is non-zero. In such case, the matrix

$$\begin{bmatrix} 1 & 3 & 7 \\ 3 & -1 & 11 \\ 1 & -1 & a \end{bmatrix}$$

is invertible, and the unique solution is given by

$$\begin{bmatrix} 1 & 3 & 7 \\ 3 & -1 & 11 \\ 1 & -1 & a \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Hence a is any real numbers except 3.

4(a). Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in $\mathbf{R}^5$. Prove that there is a nonzero vector $\mathbf{x}$ that is perpendicular to each of those vectors.

Solution: If $\mathbf{x}$ is orthogonal to $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, then it must satisfy the system

$$\mathbf{v}_1 \cdot \mathbf{x} = \mathbf{v}_2 \cdot \mathbf{x} = \mathbf{v}_3 \cdot \mathbf{x} = 0.$$

Equivalently $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{bmatrix}.$$

A is a 3×5 matrix, and the rank of A is at most 3. By rank-nullity relation, the nullity of A must be at least two. Therefore $N(A)$ is non-trivial, and there is at least one non-zero vector orthogonal to $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

4(b). Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are nonzero vectors that are orthogonal to each other. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution: Suppose a_1, a_2 and a_3 are scalars such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}.$$

Taking dot product on both sides by $\mathbf{v}_1$, we have

$$a_1\mathbf{v}_1 \cdot \mathbf{v}_1 + a_2\mathbf{v}_2 \cdot \mathbf{v}_1 + a_3\mathbf{v}_3 \cdot \mathbf{v}_1 = 0$$
$$a_1\|\mathbf{v}_1\|^2 = 0.$$

Since $\mathbf{v}_1$ is not the zero vector, $a_1 = 0$. Similarly, we can show that $a_2 = a_3 = 0$. Therefore, all coefficients a_1, a_2 and a_3 are zeros. This implies that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent.

5. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbf{R}^3$, and suppose $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation such that

$$T(\mathbf{v}_1) = 2\mathbf{v}_3 \qquad T(\mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_3 \qquad T(\mathbf{v}_3) = 10\mathbf{v}_1.$$

5(a). Find the matrix B for T with respect to the basis $\mathcal{B}$.

Solution. The matrix B has columns $[T(\mathbf{v}_1)]_{\mathcal{B}}$, $[T(\mathbf{v}_2)]_{\mathcal{B}}$, and $[T(\mathbf{v}_3)]_{\mathcal{B}}$. Therefore

$$B = \begin{bmatrix} 0 & 1 & 10 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

5(b). Suppose that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the matrix A for T with respect to the standard basis for $\mathbf{R}^3$.

[Hint: You may use your answer to **5(a)**. However, it is easier to find A directly, without using the matrix B.]

Solution. The matrix A has columns $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$. But notice that $\mathbf{e}_1 = \mathbf{v}_1$, $\mathbf{e}_2 = \mathbf{v}_2 - \mathbf{v}_1$, and $\mathbf{e}_3 = \mathbf{v}_3 - \mathbf{v}_2$. So

$$T(\mathbf{e}_1) = T(\mathbf{v}_1) = 2\mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$T(\mathbf{e}_2) = T(\mathbf{v}_2 - \mathbf{v}_1) = T(\mathbf{v}_2) - T(\mathbf{v}_1) = \mathbf{v}_1 - \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$T(\mathbf{e}_3) = T(\mathbf{v}_3 - \mathbf{v}_2) = T(\mathbf{v}_3) - T(\mathbf{v}_2) = 9\mathbf{v}_1 - \mathbf{v}_3 = \begin{bmatrix} 8 \\ -1 \\ -1 \end{bmatrix}$$

Therefore

$$A = \begin{bmatrix} 2 & 0 & 8 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

(We could have also solved for A using the formula $A = CBC^{-1}$, where

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is the change-of-basis matrix (with columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$). The matrix C^{-1} is found by row-reduction similarly to in the following problem.)

6. Find A^{-1}, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

Solution: We find A^{-1} by the formula $\text{rref}([A|I_3]) = [I_3|A^{-1}]$. (This, of course, assumes that A is invertible.) So our job is to row reduce $[A|I_3]$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 \to R3 \to R2 \to R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R3-R2]{R2-R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

Therefore

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

7(a). Find the angle between the vectors $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

Solution: According to the definition of the dot product, the cosine of the angle θ between $\mathbf{u}$ and $\mathbf{v}$ is:

$$\cos(\theta) = \frac{\mathbf{u}.\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}$$

This gives here:

$$\cos(\theta) = \frac{-8 - 2 + 1}{\sqrt{16 + 1 + 1}\sqrt{4 + 4 + 1}} = \frac{-1}{\sqrt{2}}$$

So depending on your choice of convention on the sign of θ (both are acceptable),

$$\theta = \pm\frac{3\pi}{4}$$

(two points for giving the correct relation between dot product and cosine, 5 points as soon as the cosine is correct)

7(b). Find the area of the triangle with vertices $(4, -1, 1)$, $(-2, 2, 1)$, and $(0, 0, 0)$.

Solution: The area of the triangle is (same notations as in (a)):

$$\mathcal{A} = \frac{1}{2}||\mathbf{u}||\,||\mathbf{v}||\,|\sin(\theta)|$$

but, $\sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$ so this gives here:

$$\mathcal{A} = \frac{1}{2}\sqrt{18}\sqrt{9}\sqrt{1 - \frac{1}{2}} = 4.5$$

(one point to give the formula for the area of the triangle, two for giving the one with the sine of the angle, 5 points for the correct answer.)

8. Let $f(x,y)$ be the temperature at point (x,y), and suppose that

$$\frac{\partial f}{\partial x}(1,2)=3 \qquad \frac{\partial f}{\partial y}(1,2)=4.$$

8(a). Find the directional derivative of f at $(1,2)$ in the direction $\mathbf{v}=\begin{bmatrix}1\\1\end{bmatrix}$.

Solution: According to definition, the directional derivative in direction $\mathbf{v}$ is:

$$D_{\mathbf{v}}f(1,2)=\nabla f(1,2).\frac{\mathbf{v}}{||\mathbf{v}||}$$

That gives here:

$$D_{\mathbf{v}}f(1,2)=\frac{1}{\sqrt{2}}(3+4)=\frac{7}{\sqrt{2}}$$

(one point to give the correct formula with the normalized vector, 5 points for the correct answer.)

8(b). An insect crawls along an isotherm (i.e., a level set of f) with speed 3. At time $t=0$, the insect is at the point $(1,2)$ and its x-coordinate is increasing. Find its velocity at time 0.

Solution: Let the position of the insect at time $t=0$ be $\mathbf{x}_0=\begin{bmatrix}x_0\\y_0\end{bmatrix}$ and its velocity at this time be $\mathbf{v}_0=\begin{bmatrix}v_1\\v_2\end{bmatrix}$.

We know that because the insect crawls onto a level set of f, its velocity is perpendicular to the gradient of f :

$$\nabla f(\mathbf{x}_0).\mathbf{v}_0=0$$

which gives us:

$$3v_1+4v_2=0$$

More over, we know that the speed is 3, so:

$$v_1^2 + v_2^2 = 9$$

This gives:

$$v_1^2\,(1 + (-\frac{3}{4})^2) = 9$$

$$\Leftrightarrow \quad v_1^2(\frac{25}{16}) = 9$$

Which is, knowing that $v_1 > 0$:

$$v_1 = \frac{12}{5}, \quad v_2 = -\frac{9}{5}$$

(one point for each correct condition (speed and orthogonality with the gradient) 5 points for the correct answer, "close" answers get between 3 and 5 points.)

9(a). Let $f(x,y) = \sin(xy) + xy^2$. Find the linear approximation $L(x,y)$ to $f(x,y)$ at the point $(\pi, 3)$.

Solution: Recall that the linear approximation to a function $f(x,y)$ at the point (x_0, y_0) is given by the equation

$$L(x,y) = f(x_0,y_0) + \mathcal{J}_f(x_0,y_0)(x - x_0, y - y_0)^T.$$

In our curruent situation $f(x,y) = \sin(xy) + xy^2$ and $(x_0, y_0) = (\pi, 3)$.

So we have

$$\mathcal{J}_f = \begin{pmatrix} y\cos(xy) + y^2 & x\cos(xy) + 2xy \end{pmatrix}$$

and at the point $(\pi, 3)$ this is $(3\cos 3\pi + 9, \pi\cos(3\pi) + 2\cdot\pi\cdot 3) = (6, 5\pi)$.

And $f(x_0, y_0) = \sin(3\pi) + \pi 3^2 = 9\pi$.

Thus the final expression for the linear approximation

$$\begin{aligned} L(x,y) &= 9\pi + (6 \;\; 5\pi)\begin{pmatrix} x-\pi \\ y-3 \end{pmatrix} = 9\pi + 6(x-\pi) + 5\pi(y-3) \\ &= 9\pi + 6x - 6\pi + 5\pi y - 15\pi = 6x + 5\pi y - 12\pi \end{aligned}$$

Common Errors: For the first part common errors include : completely botching the formula for linear approximation. Also, every single one of you shoudl know what $\sin 3\pi$ and $\cos 3\pi$ are. I would guess that almost half of the solutions turned in failed to actually compute those.

9(b). Suppose that $g : \mathbf{R}^2 \to \mathbf{R}^2$,

$$g(1,2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \frac{\partial g}{\partial x}(1,2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \frac{\partial g}{\partial y}(1,2) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Use this information to find (x, y) so that $g(x, y)$ should be approximately equal to $\begin{bmatrix} 1.005 \\ 1.006 \end{bmatrix}$.

Solution: Using the information in the problem we can write down an approximation to g at the point $(1, 2)$:

$$L_g(x,y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}.$$

We want (x, y) such that the approxmation is (1.005, 1006), so we solve

$$\begin{pmatrix} 1.005 \\ 1.006 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}.$$

sutract $(1, 1)^T$ from both sides to get

$$\begin{pmatrix} .005 \\ .006 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$$

Multiply both sides by the inverse of $\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ (which you should be able to compute immediately)

$$\begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} .005 \\ .006 \end{pmatrix} = \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}.$$

Compute the left hand side and get $(.002, .001)^T$. Thus

$$\begin{pmatrix} .002 \\ .001 \end{pmatrix} = \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$$

So we easily get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.002 \\ 2.001 \end{pmatrix}$$

Common Errors: For the second part the main errors (I would sy 80%) were arithmetic. This is partially because *no one* solved it by just using linear algebra techniques (which make it so that there are few computations involved). Pretty much everyone got to the equation to solve, but then they multiplied out the matrices and solved a 2 equation in 2 unknown system of equations involving lots of decimal places.

10. Suppose that A is a matrix whose row reduced echelon form is

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

10(a). Find a basis for the nullspace $N(A)$ of A.

Solution: $(x_1, x_2, x_3, x_4, x_5)^T$ is in the null space of A if and only if it is in the null space of the rref of A. And this vector will be in the null space of the rref of A if

$$\begin{pmatrix} x_1 - 2x_3 + x_4 \\ x_2 - x_3 + 3x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now solve for the pivot variables in terms of the free variables. We get $x_1 = 2x_3 - x_4$ and $x_2 = x_3 - 3x_4$ and $x_5 = 0$ .his means we can write our vector $(x_1, x_2, x_3, x_4, x_5)$ as

$$\begin{pmatrix} 2x_3 - x_4 \\ x_3 - 3x_4 \\ x_3 \\ x_4 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

So these two vectors represent a basis.

10(b). Let $\mathbf{c} = A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$. Find all solutions of $A\mathbf{x} = \mathbf{c}$.

Solution: $(1, 2, 3, 4, 5)^T$ is a very obvious solution, and one knows from general theory that the other solutions lie in the null space. So $\mathbf{x} = (1, 2, 3, 4, 5)^T + s\vec{v}_1 + t\vec{v}_2$ where $\vec{v}_1$ and $\vec{v}_2$ are the vectors above.

10(c). Is there a $\mathbf{b} \in \mathbf{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ has no solutions? Explain.

Solution: Nope. There is no such $\mathbf{b}$. There is a pivot in ever row, so the matrix A is surjective, so any $\mathbf{b}$ can be attainted.

10(d). Is there a $\mathbf{b} \in \mathbf{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ has exactly one solution? Explain.

Solution: Nope. If we had a solution we could add any element of the nullspace to it to get another solution.

11. The position of a particle at time t is $\mathbf{u}(t) = (2t, t^2, t^3/3)$.

11(a). Find the velocity of the particle at time t.

Solution: The position $\mathbf{u}(t)$ is a differentiable function of the t variable, as all its coordinate functions are polynomials in t, hence differentiable. Therefore, the velocity is given by:

$$\boxed{\mathbf{v}(t) = \mathbf{u}'(t) = (2, 2t, t^2).}$$

11(b). Find an equation for the plane P that intersects the particle's path orthogonally at the point $(0, 1, 0)$.

The problem is not correct. Don't do it.

11(c). Find the length of the particle's path from $t = 0$ to $t = 7$.

Solution: Let's calculate the length of the velocity vector at the time t:

$$\|\mathbf{v}(t)\| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2 + 2)^2} = t^2 + 2$$

as t^2+2 is ≥ 0. This is a continuous function in the variable t, therefore the sought arc length is:

$$\ell(0, 7) = \int_0^7 \|\mathbf{v}(t)\| dt = \int_0^7 (t^2 + 2) dt = \boxed{\frac{7^3}{3} + 14.}$$

Comments:

- The arc length was only seen in two exercises, and math 51 is not about integration, therefore (a) was out of 7pts, and (c) out of 3pts: very few students remembered the formula of the arc length, which comforted the grader in his choice.

- An answer containing (even strictly) the correct one was fully credited. This means that the possible contradictions have not been taken into account.
- The absence of the crucial hypotheses "*differentiable, continuous*" was not penalized at all. In particular, writing "$(2, 2t, t^2)$" in (a) gave 7 points.
- In (a), 5pts if one component was incorrect (usually the 3rd), 3pts if two were false (this includes an answer lying in $\mathbb{R}^4$).
- In (b), many computed the distance between $\mathbf{u}(0)$ and $\mathbf{u}(7)$: this led to 0pts; but writing the symbol $\int_0^7$ gave 1 point, even if the function inside was the wrong one, and 2 points if one wrote the correct integral, in the form $\int_0^7 \|\mathbf{v}(t)\| dt$ or $\int_0^7 \sqrt{4 + 4t^2 + t^4} dt$. 3 points if the integral was computed, or in case of an small mistake of computation when computing 7^3, etc.

12. The position of a particle at time t is $\mathbf{u}(t) = (x(t), y(t))$. Let $r(t)$ and $\theta(t)$ be the polar coordinates of the particle's position at time t (so that $x = r\cos\theta$ and $y = r\sin\theta$). Suppose that $r(0) = 5$ and $\theta(0) = \pi/3$. Find the particle's velocity $\mathbf{u}'(0)$ in terms of $r'(0)$ and $\theta'(0)$. (Your answer should be an expression involving $r'(0)$ and $\theta'(0)$.)

Solution: Writing x and y as functions of r and θ, we have

$$\mathbf{u}'(t) = \frac{d}{dt}\begin{bmatrix} x(r(t),\theta(t)) \\ y(r(t),\theta(t)) \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} \\ \frac{\partial y}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial \theta}\frac{d\theta}{dt} \end{bmatrix}$$

Since $x(r,\theta) = r\cos\theta$ and $y(r,\theta) = r\sin\theta$,

$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta,$$
$$\frac{\partial y}{\partial r} = \sin\theta, \qquad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Applying these partial derivatives to the chain rule expression gives

$$\mathbf{u}'(t) = \begin{bmatrix} \cos\theta(t) r'(t) - r(t)\sin\theta(t)\theta'(t) \\ \sin\theta(t) r'(t) + r(t)\cos\theta(t)\theta'(t) \end{bmatrix}$$

At $t = 0$, since $r(0) = 5$ and $\theta(0) = \pi/3$,

$$\mathbf{u}'(0) = \begin{bmatrix} \frac{1}{2}r'(0) - \frac{5\sqrt{3}}{2}\theta'(0) \\ \frac{\sqrt{3}}{2}r'(0) + \frac{5}{2}\theta'(0) \end{bmatrix},$$

using that $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Notes on grading and common errors:

- For the chain rule, up to 4 points total were awarded, depending on how close the student came to writing down a form of the chain rule from which it would have been possible to obtain the right answer. Many students wrote $\mathbf{u}(t)$ as a composite of a function $\mathbf{u}(x, y)$ and the x, y, r, θ functions above. In this case, though, $\mathbf{u}(x, y) = (x, y)$, so $\mathbf{u}$ is the identity function. In any case, applying the chain rule gives

  $$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial x}\left(\frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt}\right) + \frac{\partial \mathbf{u}}{\partial y}\left(\frac{\partial y}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial \theta}\frac{d\theta}{dt}\right)$$

  This was not immediately marked as wrong, although very few students subsequently evaluated the partial derivatives $\frac{\partial \mathbf{u}}{\partial x}$ and $\frac{\partial \mathbf{u}}{\partial y}$ correctly (see below).

  Using Jacobian matrices for this step was also acceptable and received full credit if done correctly.

- Two more points were awarded for the correct calculation of the partial derivatives in the chain rule expression, typically including the derivatives of x and y noted above. If the function $\mathbf{u}(x, y) = (x, y)$ was used, then $D\mathbf{u} = I_2$, so the partial derivatives of $\mathbf{u}$ are
$$\frac{\partial \mathbf{u}}{\partial x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \frac{\partial \mathbf{u}}{\partial y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
Many students miscomputed these as 1 or as some more complicated scalar, leading to a real-valued answer for $\mathbf{u}'(0)$. Since $\mathbf{u}(t)$ takes values in $\mathbf{R}^2$, though, $\mathbf{u}'(0)$ must be a vector in $\mathbf{R}^2$.
- An additional point was awarded for the correct substitution of these derivatives into the chain rule expression.
- A total of 3 points were awarded for correctly evaluating the chain rule expression at $(r, \theta) = (5, \pi/3)$. Many students miscomputed the values of $\cos \pi/3$ and $\sin \pi/3$. One point was deducted for leaving these trigonometric expressions unevaluated.
- Since the question required only for the student to find the value of $\mathbf{u}'(0)$, credit was given if these steps were done implicitly. Particularly in partial-credit cases, additional points may have been deducted for lack of clarity, at the discretion of the grader.

13(a). Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution: We compute out the characteristic polynomial of A to be

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{bmatrix} \\ &= (\lambda - 1)(\lambda - 4) - (-2)(-3) \\ &= \lambda^2 - 5\lambda + 4 - 6 \\ &= \lambda^2 - 5\lambda - 2\,. \end{aligned}$$

The eigenvalues of A are the roots of this equation; by the quadratic equation, they are

$$\lambda = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-2)}}{2} = \frac{5 \pm \sqrt{33}}{2}\,.$$

13(b). Find an eigenvector with eigenvalue $\lambda = 3$ for the matrix

$$M = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix}.$$

Solution: An eigenvector with eigenvalue $\lambda = 3$ for the matrix M is simply a non-zero vector in the nullspace of

$$3I - M = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & 1 \\ -2 & 4 & 2 \end{bmatrix}.$$

Notice that the second and third rows (of $3I - M$) are scalar multiples of the first row; hence, $3I - M$ row reduces to

$$\text{rref}(3I - M) = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the 3-eigenspace of A is the span of the vectors $(2, 1, 0)$ and $(1, 0, 1)$.

14(a). Let $f(x,y) = 3x - x^3 - 3xy^2$. Find the Hessian $H_f(x,y)$ of f at (x,y).

Solution: We take first-order derivatives to compute the Jacobian:

$$D_f(x,y) = \begin{bmatrix} 3 - 3x^2 - 3y^2 & -6xy \end{bmatrix}.$$

Taking second-order derivatives, we compute the Hessian:

$$H_f(x,y) = \begin{bmatrix} -6x & -6y \\ -6y & -6x \end{bmatrix}.$$

14(b). The function f (in part (a)) has critical points $(1,0)$, $(-1,0)$, $(0,1)$, and $(0,-1)$. Determine whether $(1,0)$ is a local maximum, a local minimum, or a saddle point.

Solution: At the point $(1,0)$, the Hessian evaluates to:

$$H_f(1,0) = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}.$$

We wish to check the definiteness of this matrix. One valid way is to compute the eigenvalues of this matrix and then check their signs; however, for 2×2 matrices, it is sufficient to look at the signs of the determinant and trace.

By the first method, it is readily seen that $H_f(1,0)$ has one eigenvalue, -6, of multiplicity 2 — hence, $H_f(1,0)$ is negative definite and $(1,0)$ is a local maximum for f.

Alternatively, $H_f(1,0)$ has determinant $36 > 0$ and trace $-12 < 0$, so this also shows that $H_f(1,0)$ is negative definite.

14(c). Determine whether $(0,1)$ is a local maximum, a local minimum, or a saddle point.

Solution: Similarly, at the point $(0,1)$, the Hessian evaluates to:

$$H_f(0,1) = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}.$$

Here, it is not so obvious what the eigenvalues of $H_f(0,1)$ are, so we resort to the trace/determinant method. Since $H_f(0,1)$ has determinant $-36 < 0$, it is indefinite (and the trace doesn't matter); hence, $(0,1)$ is a saddle point for f.

15. Suppose $F : \mathbf{R}^2 \to \mathbf{R}^3$ is defined by $F(x,y) = \begin{bmatrix} x^2y + \sin y \\ e^{7y} \\ 2x + 3y^2 \end{bmatrix}$.

Find the Jacobian matrix (i.e, the matrix for the total derivative) $DF(x,y)$.

Solution: Let us write

$$F(x,y) = \begin{bmatrix} f_1(x,y) \\ \\ f_2(x,y) \\ \\ f_3(x,y) \end{bmatrix}$$

where $f_1(x,y) = x^2y + \sin y$, $f_2(x,y) = e^{7y}$ and $f_3(x,y) = 2x + 3xy^2$. By the definition of jacobian we have

$$\boxed{DF(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 + \cos y \\ \\ 0 & 7e^{7y} \\ \\ 2 & 6y \end{bmatrix}}$$

Comments: Mistakes made by students come roughly in two flavors

- Computing the partial derivatives incorrectly. This error comes from not remembering the derivatives of functions like $\sin y$, or when one does not understand the meaning of $\frac{\partial f}{\partial x_i}$ for a function $f(x_1,\ldots,x_n)$. Remember that when one computes $\frac{\partial f}{\partial x_i}$, one regards x_i as the variable, thinks of the other x_j $(i \neq j)$ as constants, and calculates the derivative as for one variable functions.
- Writing the transpose of the Jacobian, instead of the matrix itself. A useful thing to remember is that the i-th column of the Jacobian represents the derivative with respect to the variable x_i, of all the functions that make up F. Thus, if $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ then its Jacobian should be an $m \times n$ matrix.

16. In the following sentences, A is an $n \times n$ matrix and R is its reduced row echelon form. For each sentence, circle **T** if it is always true, **F** if it is always false, and **S** if it is sometimes true and sometimes false (i.e, if more information is needed to determine whether it is true or false). No explanations are necessary.

(1) If $\mathbf{b}$ is in $C(A)$, then $A\mathbf{x} = \mathbf{b}$ has a solution.

Solution: $\boxed{\textbf{T}}$ Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the columns of the matrix A. If $\mathbf{b} \in C(A)$, then $\mathbf{b} \in span(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and there exist $c_1, \ldots, c_n \in \mathbf{R}$ so that $\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$. From the definition of multiplication of a matrix by a vector we get

$$A \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{b}$$

which is exactly a solution to the system $A\mathbf{x} = \mathbf{b}$.

(2) If $N(A) = \{\mathbf{0}\}$, then $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

Solution: $\boxed{\textbf{T}}$ Since nullity of A is zero then whenever the system is solvable (consistent), the solution must be unique. By the rank-nullity theorem we have $C(A) = \mathbf{R}^n$ and by part 1 the system has a solution for any $\mathbf{b} \in \mathbf{R}^n$.

(3) If R has a pivot in every row then A is invertible.

Solution: $\boxed{\textbf{T}}$ Recall that a linear transformation is invertible if and only if it is both injective and surjective. In the case of matrices this is the same as having $null(A) = 0$ and $rank(A) = n$. Since A is $n \times n$, the rank-nullity theorem implies that A will be invertible whenever $null(A) = 0$ or $rank(A) = n$ (since one follows from the other). Since R has a pivot in every column, $rank(A) = n$ and A is invertible.

(4) If a column of R has no pivot, then 0 is an eigenvalue of A.

Solution: $\boxed{\textbf{T}}$ If a column of R has no pivot then $rank(A) < n$ and by the rank-nullity theorem $null(A) \geq 1$. That is, there exists a nonzero vector $\mathbf{v} \in \mathbf{R}^n$ so that $A\mathbf{v} = \mathbf{0} = 0\mathbf{v}$ and therefore 0 is an eigenvalue.

(5) If A is symmetric, then A is diagonalizable.

Solution: $\boxed{\textbf{T}}$ This is exactly the content of the spectral theorem.

(6) If 3 vectors in $\mathbf{R}^5$ are linearly independent, then the matrix whose columns are these 3 vectors is an invertible matrix.

Solution: $\boxed{\textbf{F}}$ This is always false since the matrix will not be square.

(7) If n vectors in $\mathbf{R}^n$ are linearly independent, then the matrix whose columns are these n vectors is invertible.

Solution: $\boxed{\mathbf{T}}$ Let A be the resulting matrix, the it follows that A is $n \times n$. In this case A is invertible if and only if it $null(A) = 0$ or $rank(A) = n$ (see part 3). Finally, recall that a matrix has linearly independent columns if and only if its nullity is zero.

(8) If a differentiable function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ has a minimum at $\mathbf{x}$, then $\mathbf{x}$ is a critical point of f.

Solution: $\boxed{\mathbf{T}}$ If $\mathbf{x}$ is a local minimum for f then $\nabla f(\mathbf{x}) = \mathbf{0}$, which is the definition of a critical point.

(9) If $\mathbf{x}$ is a critical point of $f : \mathbf{R}^n \longrightarrow \mathbf{R}$, then f has a local maximum or a local minimum at $\mathbf{x}$.

Solution: $\boxed{\mathbf{F} \text{ or } \mathbf{S}}$ In this question depending on the interpretation, one could get either answer so both were considered to be correct. If $\mathbf{x}$ is a critical point then some times it is a local minimum, others a local maximum or some other times a saddle point. Thus, **S** is a possible answer. On the other hand, if you look at the sentence as a logical statement

$$p \text{ implies } q$$

and one notices that (every time p happens, then one has q) is false in this case, then **F** comes also as a possibility.

(10) If Q is a positive definite quadratic form, then its associated symmetric matrix is invertible.

Solution: $\boxed{\mathbf{T}}$ There are at least two ways of solving this and both rely on the fact that a quadratic form is positive definite if and only if all the eigenvalues of its associated matrix are positive. The first method uses that since the determinant is also the product of the eigenvalues then it is not zero and therefore the matrix is invertible. The second method is by noticing that a square matrix is invertible if and only if zero is not an eigenvalue. Since all the eigenvalues are positive, the matrix is invertible.