# Math 51 - Spring 2010 - Midterm Exam II

**Problem 1.** (10 pts.) Let $P$ be the quadrilateral in $\mathbb{R}^2$ with vertices $(0,0)$, $(2,0)$, $(3,3)$, and $(-1,1)$

a) Find the area of the quadrilateral $P$.
**Solution:** (6 points) Let points $A = (0,0), B = (2,0), C = (3,3), D = (-1,1)$. The area for triangle $ABC$ is

$$\frac{1}{2}\left|\det\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}\right| = 3.$$

And the area for triangle $ACD$ is

$$\frac{1}{2}\left|\det\begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix}\right| = 3.$$

Therefore the are of the quadrilateral $P$ is the area for triangle $ABC$ + the area for triangle $ACD$ which is 6.

**Comments:** Common mistakes is to say that the area of the quadrilateral is

$$\left|\det\begin{bmatrix} \vec{v_1} \\ \vec{v_2} \end{bmatrix}\right|,$$

where $\vec{v_1}, \vec{v_2}$ are some vectors from points $A, B, C, D$. If your answer happens to be correct, but your method is wrong. I don't give any partial credit.

b) Let $A = \begin{bmatrix} 172 & 2 \\ 1 & 0 \end{bmatrix}$ and $T$ be the corresponding linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$. Find the area of $T(P)$.
**Solution:** The area of $T(P) =$ (the area of $P$) $\times |\det(A)| = 6 \cdot \left|\det\begin{bmatrix} 172 & 2 \\ 1 & 0 \end{bmatrix}\right| = 12.$

**Comments:** A common mistake is to say the area of $T(P)$ is $|\det(A)|$. Also full credit is granted if you use the formula T(P) above but have the wrong area of P from part (a).

**Problem 2.** (15 pts.) Let $T$ be the orthogonal projection of $\mathbb{R}^2$ onto the line $y = -x$ and $A$ be its matrix with respect to the standard basis.

a) Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ consisting of eigenvectors of $A$.
**Solution:** The line $y = -x$ is spanned by a vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Since all points on the line $y = -x$ are fixed by the projection, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue 1. The perpendicular direction, spanned by a vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is smashed to zero, i.e. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 0. Thus $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

**Comments:** Some students remembered the formula for the matrix of projection in standard coordinates. From this one could find its eigenvalues and eigenvectors. This "sweaty" solution – if done correctly – still could give you full credit, but it was not the intended approach to this problem.

b) Find the matrix of the linear map $T$ with respect to the eigenbasis $\mathcal{B}$ described in part a).
**Solution:** $B$ is the diagonal matrix with eigenvalues on the diagonal, i.e. $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. Just be careful to put the eigenvalues on the diagonal in the same order as you put the eigenvectors in $\mathcal{B}$.

c) Either find $A^{-1}$ or show that $A$ is not invertible.
**Solution:** As $A$ and $B$ are similar, $\det A = \det B = 0$. Thus neither $A$ nor $B$ is invertible.

**Comments:** Showing that $\det B = 0$ is not enough, as the question asked about invertibility of $A$. Some students commented that $\det A = 0$, hence $A$ is not diagonalizable. We should exploit this misunderstanding in the future exam.

**Problem 3.** (10 pts.) Let $A$ be a $3 \times 3$ matrix and $\mathbf{v}$ and $\mathbf{w}$ be two vectors in $\mathbb{R}^3$ such that

$$A\mathbf{v} = \mathbf{v} \quad \text{and} \quad A\mathbf{w} = -\mathbf{w}$$

a) Explain why $\mathbf{v}$ and $\mathbf{w}$ must be linearly independent.

You can quote a theorem from the course or prove it directly.
**Solution:** There were two ways to do this. One was to first observe that $\mathbf{v}$ was an eigenvector with eigenvalue 1, and $\mathbf{w}$ was an eigenvector with eigenvalue $-1$. Hence, from a result in class, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent because they lie in different eigenspaces. The other method that was accepted was to just prove directly that they were linearly independent. That is, suppose you have a relation

$$\lambda\mathbf{v} + \mu\mathbf{w} = 0,$$

then

$$0 = A(\lambda\mathbf{v} + \mu\mathbf{w}) = \lambda A\mathbf{v} + \mu A\mathbf{w} = \lambda\mathbf{v} - \mu\mathbf{w}.$$

By adding these two sets of equations together you obtain (since we're assuming $\mathbf{v} \neq 0$):

$$2\lambda\mathbf{v} = 0 \implies \lambda = 0.$$

One concludes from this (since $\mathbf{w} \neq 0$) that $\mu = 0$ and so $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

b) Show that for the above vectors $\mathbf{v} + \mathbf{w}$ is not an eigenvector of $A$.

(In one of the ways to do it: you can assume that $\mathbf{v} + \mathbf{w}$ is an eigenvector of $A$ and conclude that $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent in contradiction to part a.)
**Solution:** The easiest way was to follow the hint, and it was typically quite difficult to get it completely correct by not approximately following this solution. First, suppose

that $\mathbf{v}+\mathbf{w}$ is an eigenvector of $A$, then $A(\mathbf{v}+\mathbf{w}) = \lambda(\mathbf{v}+\mathbf{w})$ for some $\lambda$. Now, expanding this out in both directions gives the relation:

$$\mathbf{v}-\mathbf{w} = A\mathbf{v}+A\mathbf{w} = A(\mathbf{v}+\mathbf{w}) = \lambda(\mathbf{v}+\mathbf{w}) = \lambda\mathbf{v}+\lambda\mathbf{w}.$$

Rearranging everything on the LHS gives the equation:

$$(1-\lambda)\mathbf{v}+(-1-\lambda)\mathbf{w} = 0.$$

But we showed (a) that $\mathbf{v}$ and $\mathbf{w}$ were LI, thus we conclude that $1-\lambda = 1+\lambda = 0$, which is impossible.

**Problem 4.** (10 pts.)

a) Let $A$ be a $3\times 3$ matrix such that

$$A\begin{bmatrix}1\\-4\\2\end{bmatrix} = \mathbf{e}_1,\quad A\begin{bmatrix}13\\-6\\0\end{bmatrix} = \mathbf{e}_2 \text{ and } A\begin{bmatrix}31\\-12\\10\end{bmatrix} = \mathbf{e}_3$$

Find $A^{-1}$.
**Solution.** By hypothesis,

$$A\begin{bmatrix}1 & 13 & 31\\-4 & -6 & -12\\2 & 0 & 10\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I_3.$$

Thus $\begin{bmatrix}1 & 13 & 31\\-4 & -6 & -12\\2 & 0 & 10\end{bmatrix} = A^{-1}.$

b) Show that if 5 is an eigenvalue of an invertible matrix $B$, then $\dfrac{1}{5}$ is an eigenvalue of the matrix $B^{-1}$.
**Solution.** (5 points) If 5 is an eigenvalue of an invertible matrix $B$, then there is an eigenvector $v$ such that $Bv = 5v$. Since $B$ is invertible,

$$\begin{aligned}B^{-1}(Bv) &= B^{-1}(5v)\\ v &= 5B^{-1}v\\ \frac{1}{5}v &= B^{-1}v\end{aligned}$$

So $\frac{1}{5}$ is an eigenvalue of $B^{-1}$.
**Comments:** Some students said that since $det(B^{-1}) = \frac{1}{det(B)}$ and $det(B)$ is the product of all eigenvalues $\lambda_i$ of $B$, $det(B^{-1})$ is the product of $\frac{1}{\lambda_i}$, so every eigenvalue of $B^{-1}$ should be $\frac{1}{\lambda_i}$. This proof makes no sense. For example, if $B$ is $2\times 2$ and two eigenvalues are 4 and 5, then $det(B^{-1}) = \frac{1}{20}$. The argument above doesn't show why the eigenvalues of $B^{-1}$ can't be $\frac{1}{2}$ and $\frac{1}{10}$.

**Problem 5.** (15 pts.) Suppose $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 3 & \alpha \end{bmatrix}$ for some unknown $\alpha \in \mathbb{R}$.

a) Find the characteristic polynomial of $A$. (It will contain $\alpha$.)
**Solution:** The characteristic polynomial $p(\lambda)$ of $A$ is

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & -2 \\ 0 & \lambda + 1 & 0 \\ 0 & -3 & \lambda - \alpha \end{vmatrix} = (\lambda - 1) \begin{vmatrix} \lambda + 1 & 0 \\ -3 & \lambda - \alpha \end{vmatrix} = (\lambda - 1)(\lambda + 1)(\lambda - \alpha).$$

**Comments:** Some students confuse the characteristic polynomial with the polynomial associated to a symmetric matrix representing a quadratic form. Some students row-reduce the matrix first, and then find the characteristic polynomial of the reduced matrix. This is, in general, not equal to that of the unreduced one.

b) Show that for any $|\alpha| \neq 1$ the matrix $A$ is diagonalizable.
**Solution:** If $|\alpha| \neq 1$, then the $3 \times 3$ matrix $A$ has three distinct eigenvalues, which implies that $A$ is diagonalizable. (Theorem: Any $n \times n$ matrix which has $n$ distinct eigenvalues is diagonalizable.)

**Comments:** Many students stated that "an $n \times n$ matrix is diagonalizable if and only if it has $n$ distinct eigenvalues", which is wrong. Some students proved for a specific $a$, say $a = 2$, others showed that it is not diagonalizable for $a = 1$, which has nothing to do with $|a| \neq 1$.

c) Show that for $\alpha = 1$ the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 3 & \alpha \end{bmatrix}$ is not diagonalizable.

**Problem 6.** (10 pts.) For what real numbers $a$, $b$, and $c$ does the function $f(x, y) = ax^2 + bxy + cy^2$ satisfy the partial differential equation

$$f_{xx} + f_{yy} = 0?$$

**Solution:**

$$\begin{aligned} f_x &= 2ax + by \\ f_{xx} &= 2a \\ f_x &= 2cy + bx \\ f_{xx} &= 2c \\ 0 = f_{xx} + f_{yy} &= 2a + 2c \end{aligned}$$

So $a = -c$ and $b$ can be anything.

**Problem 7.** (10 pts.) Let $\mathbf{r}(t) = (t^3 + t - 3,\ 2t^2 - 1)$ for $t \in \mathbb{R}$ and let $\mathcal{C}$ be the image curve. Find an equation of the tangent line to $\mathcal{C}$ at the point $(-1, 1)$.
**Solution:**

$$\mathbf{r}'(t) = (3t^2, 2t^2 - 1)$$

$$\mathbf{r}(1) = (1 + 1 - 3, 2 - 1) = (-1, 1)$$

so 1 is the time when the curve passes through $(-1, 1)$.

$$\mathbf{r}'(1) = (4, 4)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{4} = 1$$

$$y - 1 = 1(x + 1)$$

$$y = x + 2$$

**Comments:** Common errors included using the equation for a tangent plane instead of using the equations for lines, not realizing that one needed to solve for the time that the curve passed throught the point, and not understanding the relationship between $r'$ and slope.

**Problem 8.** (10 pts.) Let $Q(x, y) = x^2 + 2a\,xy + 4y^2$, where $a$ is a parameter.

a) Write the matrix corresponding to the quadratic form $Q$. (Your answer will contain the parameter $a$.)
**Solution:** The matrix $A$ is given by

$$A = \begin{bmatrix} 1 & a \\ a & 4 \end{bmatrix}.$$

b) For what values of the parameter $a$ is the quadratic form $Q$ positive definite?
**Solution:** We need both eigenvalues of $A$ to be positive. Alternatively, we can show that the trace and determinant of $A$ are both positive. The trace of $A$ is 5, and the determinant of $A$ is $4 - a^2$, so we need $4 - a^2 > 0$, i.e., $a^2 < 4$. In other words, we need $-2 < a < 2$. A common mistake was to forget that $a^2 < 4$ yields two conditions. Many people also wrote down an expression like

$$\sqrt{9 + 4a^2} > -5$$

and squared both sides, which is not allowed.

**Problem 9.** (10 pts.) Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by the formula

$$\mathbf{f}(x, y) = (x^2 - 3y,\ xy + 2,\ y^2)$$

a) Compute the matrix of partial derivatives $D\mathbf{f}(1,2)$.

**Solution:** If $f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$, then we compute the partial derivatives of $f$:

$$\frac{\partial f_1}{\partial x} = 2x$$
$$\frac{\partial f_1}{\partial y} = -3$$
$$\frac{\partial f_2}{\partial x} = y$$
$$\frac{\partial f_2}{\partial y} = x$$
$$\frac{\partial f_3}{\partial x} = 0$$
$$\frac{\partial f_3}{\partial y} = 2y.$$

Plugging in $(x,y) = (1,2)$ yields

$$D_f(1,2) = \begin{bmatrix} 2 & -3 \\ 2 & 1 \\ 0 & 4 \end{bmatrix}.$$

b) Compute $D\mathbf{f}(1,2) \begin{bmatrix} x \\ y \end{bmatrix}$

**Solution:** We compute

$$D\mathbf{f}(1,2) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 2 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ 2x + y \\ 4y \end{bmatrix}.$$