Problem 1. (10 pts.)

a) Let $[1, 2, 3]$ be the first row of a square matrix A and $\begin{bmatrix} a \\ -1 \\ 1 \end{bmatrix}$ be the first column of a square matrix B. Find a if you know that B is the inverse of A.

$$[1,2,3]\begin{bmatrix} a \\ -1 \\ 1 \end{bmatrix} = a-2+3 = a+1 = 1 \qquad a=0$$

b) Let $[1, 2, 3]$ be the first row of a square matrix A and $\begin{bmatrix} a \\ -1 \\ 1 \end{bmatrix}$ be the second column of a square matrix B. Find a if you know that B is the inverse of A.

$$a+1=0 \implies a=-1$$

Problem 2. (10 pts.) Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-2\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 5 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

the inverse

Problem 3. (10 pts.) Assume that A is an invertible matrix.

a) If $A\,\vec{x} = \vec{u}$, what is $A^{-1}\left(2\,\vec{u}\right)$?

$\vec{x} = A^{-1}(\vec{u})$ So: $A^{-1}(2\vec{u}) = 2A^{-1}(\vec{u}) = 2\vec{x}$

b) If $A\,\vec{x_1} = 2\,\vec{u_1}$ and $A\,\vec{x_2} = 3\,\vec{u_2}$, what is $A^{-1}\left(3\,\vec{u_1} - 8\,\vec{u_2}\right)$?

$\vec{x}_1 = 2A^{-1}(\vec{u}_1)$

$\vec{x}_2 = 3A^{-1}(\vec{u}_2)$

$\Rightarrow A^{-1}(3\vec{u}_1 - 8\vec{u}_2) =$

$= 3A^{-1}(\vec{u}_1) - 8A^{-1}(\vec{u}_2) =$

$\boxed{= \frac{3}{2}\vec{x}_1 - \frac{8}{3}\vec{x}_2}$

Problem 4. Let $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ be the following linear map:

$$\mathbf{T}\left(\begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \end{bmatrix}\right) = \begin{bmatrix} 2\vec{x}_2 - \vec{x}_3 \\ 3\vec{x}_1 - 2\vec{x}_2 \\ -2\vec{x}_1 + 2\vec{x}_2 + \vec{x}_3 \end{bmatrix}.$$

a) (4 pts.) Write the matrix A of $\mathbf{T}$ (in the standard coordinates).

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

b) (6 pts.) Given that the characteristic polynomial of the above matrix A is

$$p(t) = (1-t)(t-2)(t+4)$$

(you do not need to verify that), find a basis of eigenvectors of $\mathbf{T}$.

$\boxed{\lambda = 1}$

$$\begin{bmatrix} -1 & 2 & -1 \\ 3 & -3 & 0 \\ -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \text{free} \\ \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$E_1 = N(A - I) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

$\boxed{\lambda = 2}$

$$\begin{bmatrix} -2 & 2 & -1 \\ 3 & -4 & 0 \\ -2 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} -2 & 2 & -1 \\ 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -2 & -3 \\ 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \text{free} \\ \begin{bmatrix} 0 & -2 & -3 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$E_2 = \text{Span}\left(\begin{bmatrix} -2 \\ -3/2 \\ 1 \end{bmatrix}\right)$$

$\boxed{\lambda = -4}$

$$-\left(\begin{bmatrix} +4 & 2 & -1 \\ 3 & 2 & 0 \\ -2 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ -2 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}\right.$$

$$E_{-4} = \text{Span}\left(3\begin{bmatrix} 1 \\ -3/2 \\ 1 \end{bmatrix}\right)$$

c) (4 pts.) Is **T** diagonizable? Explain your answer.

T has eigenbase, thus is diagonizable

Problem 5. (10 pts.) Find a matrix C such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = C \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} C^{-1}$$

or explain why such matrix does not exist.

If two matrices are similar, they have the same e-values.

First matrix has e-values 1 and 4, second has double e-value 2. They are different, thus <u>no such C exists</u>

Problem 6. Let A be the following 3×3 matrix:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

with its inverse

$$A^{-1} = \begin{bmatrix} -2 & -2 & 3 \\ 1 & 1 & -1 \\ -4 & -3 & 5 \end{bmatrix}$$

(you do not need to verify that).

a) (4 pts.) Solve the system

$$A \cdot \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\vec{x} = A^{-1} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ 14 \end{bmatrix}$$

b) (4 pts.) Find a matrix M that has eigenvectors $\mathbf{v_1} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, both corresponding to the eigenvalue 0, and an eigenvector $\mathbf{v_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ corresponding to the eigenvalue 1.

since $A = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$ $\qquad MA = A \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

so:

$$M = A \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} -2 & -2 & 3 \\ 1 & 1 & -1 \\ -4 & -3 & 5 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}}_{A} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & -3 & 5 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 & 3 & -5 \\ -4 & -3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

c) (4 pts.) Find the inverse M^{-1} of the matrix M described in part b) or explain why does the inverse fail to exist.

M has 0 as e-value, thus M^{-1} doesn't exist.

d) (4 pts.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map whose matrix in standard coordinates is the matrix M described in part a). Describe the map T in terms of "projections", "reflections" and/or "rotations". Be specific as to which planes, lines or points you are using in the projection, reflection or rotation.

T is projection onto the line $\mathrm{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) = \mathrm{span}(\vec{v}_3)$.

Problem 7. (10 pts.) Find an equation of the tangent plane to the graph of the function

$$f(x,y) = 2x^3y^2 - 3y^2x + 3$$

at the point $(1,1,2)$.

$\frac{\partial f}{\partial x} = 6x^2y^2 - 3y^2$ $\qquad$ $\frac{\partial f}{\partial x}(1,1) = 3$

$\frac{\partial f}{\partial y} = 4x^3y - 6yx$ $\qquad$ $\frac{\partial f}{\partial y}(1,1) = -2$

tangent plane: $\boxed{z = 3(x-1) - 2(y-1) + 2}$

Problem 8. (10 pts.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that

$$T\left(\vec{e_1} + \vec{e_2}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad T\left(\vec{e_1} - \vec{e_2}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

a) Find $T\left(\vec{e_1}\right)$.

$$T(\vec{e_1}) = T\left(\tfrac{1}{2}(\vec{e_1} + \vec{e_2} + \vec{e_1} - \vec{e_2})\right) = \tfrac{1}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

b) Find the matrix (in standard basis) of the map T.

$$T(\vec{e_2}) = T\left(\tfrac{1}{2}(\vec{e_1} + \vec{e_2} - (\vec{e_1} - \vec{e_2}))\right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

so the matrix is: $\left[T\vec{e_1}, T\vec{e_2}\right] = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$

Problem 9. (10 pts.) Let $Q(x, y) = x^2 - 2xy + ay^2$

a) For what values of the parameter a is the quadratic form Q positive definite?
Hint, which you may ignore: write $Q(x, y)$ as $(x-y)^2 + \dots \cdot y^2$ and use the definition of being positive definite.

with the hint: $Q(x,y) = (x-y)^2 + (a-1)y^2$

positive definite $\Leftrightarrow$ $a-1>0$ or $\boxed{a>1}$

w/o hint:

Matrix: $\begin{bmatrix} 1 & -1 \\ -1 & a \end{bmatrix}$ e-values: $p(\lambda) = (1-\lambda)(a-\lambda) - 1 = \lambda^2 - (a+1)\lambda + a - 1$

both roots must be positive:

smaller root: $\lambda = \dfrac{a+1-\sqrt{(a+1)^2 - 4(a-1)}}{2} > 0$

iff: $a+1 > \sqrt{a^2+2a+1-4a+4}$

iff. $a^2+2a+1 > a^2-2a+5$

iff $4a > 4 \Leftrightarrow \boxed{a>1}$

b) For what values of the constant a does the function $Q(x, y)$ satisfy the following expression:

$$\frac{\partial^2 Q}{\partial x \partial y} = \frac{\partial^2 Q}{\partial y^2}$$

$\dfrac{\partial Q}{\partial y} = -2x + 2ay$

$\dfrac{\partial^2 Q}{\partial x \partial y} = -2$ $\qquad$ $\dfrac{\partial^2 Q}{\partial y^2} = 2a$

$-2 = 2a$

$\boxed{a = -1}$