Math 51 - Winter 2009 - Midterm Exam II

Name: ______________________________

Student ID: ___________________________

Select your section:

Penka Georgieva 02 (11:00-11:50 AM) 06 (1:15-2:05 PM)	Anssi Lahtinen 03 (11:00-11:50 AM) 11 (1:15-2:05 PM)	Man Chun Li 12 (1:15-2:05 PM) 08 (11:00-11:50 AM)	Simon Rubinstein-Salzedo 17 (1:15-2:05 PM) 21 (11:00-11:50 AM)
Aaron Smith 09 (11:00-11:50 AM) 20 (10:00-10:50 AM)	Nikola Penev 14 (1:15-2:05 PM) 24 (2:15-3:05 PM)	Eric Malm 15 (11:00-11:50 AM) 23 (1:15-2:05 PM)	Yu-jong Tzeng 51A

Signature: ____________________________

Instructions: Print your name and student ID number, print your section number and TA's name, write your signature to indicate that you accept the honor code. During the test, you may not use notes, books, calculators. Read each question carefully, and show all your work.
There are nine problems on the pages numbered from 1 to 9, with the total of 100 points. Point values are given in parentheses. You have 2 hours (until 9PM) to answer all the questions.

In the exam all vectors are columns, but sometimes we use transpose to write them horizontally.

Thus $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = [v_1, v_2, \ldots, v_k]^T$.

Similarly $\mathbf{v}^T$ is a row $[v_1, v_2, \ldots, v_k]$.

The dot product of two vectors is denoted as $\mathbf{v} \cdot \mathbf{w}$.

Problem 1. (10 pts.)

a) Let $[1, 2, 3]$ be the first row of a square matrix A and $\begin{bmatrix} a \\ -1 \\ 1 \end{bmatrix}$ be the first column of a square matrix B. Find a if you know that B is the inverse of A.

b) Let $[1, 2, 3]$ be the first row of a square matrix A and $\begin{bmatrix} a \\ -1 \\ 1 \end{bmatrix}$ be the second column of a square matrix B. Find a if you know that B is the inverse of A.

Problem 2. (10 pts.) Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3. (10 pts.) Assume that A is an invertible matrix.

a) If $A\mathbf{x} = \mathbf{u}$, what is $A^{-1}(2\mathbf{u})$?

b) If $A\mathbf{x_1} = 2\mathbf{u_1}$ and $A\mathbf{x_2} = 3\mathbf{u_2}$, what is $A^{-1}(3\mathbf{u_1} - 8\mathbf{u_2})$?

Problem 4. Let $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ be the following linear map:

$$\mathbf{T}\left(\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}\right) = \begin{bmatrix} 2\mathbf{x}_2 - \mathbf{x}_3 \\ 3\mathbf{x}_1 - 2\mathbf{x}_2 \\ -2\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3 \end{bmatrix}.$$

a) (4 pts.) Write the matrix A of $\mathbf{T}$ (in the standard coordinates).

b) (6 pts.) Given that the characteristic polynomial of the above matrix A is

$$p(t) = (1 - t)(t - 2)(t + 4)$$

(you do not need to verify that), find a basis of eigenvectors of $\mathbf{T}$.

c) (4 pts.) Is $\mathbf{T}$ diagonizable? Explain your answer.

Problem 5. (10 pts.) Find a matrix C such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = C \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} C^{-1}$$

or explain why such matrix does not exist.

Problem 6. Let A be the following 3×3 matrix:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

with its inverse

$$A^{-1} = \begin{bmatrix} -2 & -2 & 3 \\ 1 & 1 & -1 \\ -4 & -3 & 5 \end{bmatrix}$$

(you do not need to verify that).

a) (4 pts.) Solve the system

$$A \cdot \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

b) (4 pts.) Find a matrix M that has eigenvectors $\mathbf{v_1} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, both corresponding to the eigenvalue 0, and an eigenvector $\mathbf{v_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ corresponding to the eigenvalue 1.

c) (4 pts.) Find the inverse M^{-1} of the matrix M described in part b) or explain why does the inverse fail to exist.

d) (4 pts.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map whose matrix in standard coordinates is the matrix M described in part a). Describe the map T in terms of "projections", "reflections" and/or "rotations". Be specific as to which planes, lines or points you are using in the projection, reflection or rotation.

Problem 7. (10 pts.) Find an equation of the tangent plane to the graph of the function

$$f(x, y) = 2x^3y^2 - 3y^2x + 3$$

at the point $(1, 1, 2)$.

Problem 8. (10 pts.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that

$$T(\mathbf{e_1} + \mathbf{e_2}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad T(\mathbf{e_1} - \mathbf{e_2}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

a) Find $T(\mathbf{e_1})$.

b) Find the matrix (in standard basis) of the map T.

Problem 9. (10 pts.) Let $Q(x, y) = x^2 - 2xy + ay^2$

a) For what values of the parameter a is the quadratic form Q positive definite?
Hint, which you may ignore: write $Q(x, y)$ as $(x - y)^2 + \ldots \cdot y^2$ and use the definition of being positive definite.

b) For what values of the constant a does the function $Q(x, y)$ satisfy the following expression:

$$\frac{\partial^2 Q}{\partial x \partial y} = \frac{\partial^2 Q}{\partial y^2}$$

Page	Score	Maximum
1		20
2		10
3		10
4		14
5		8
6		8
7		20
8		10
Total		100