Math 51 - Winter 2009 - Final Exam

Name: ______________________________

Student ID: ______________________________

Select your section:

Penka Georgieva 02 (11:00-11:50 AM) 06 (1:15-2:05 PM)	Anssi Lahtinen 03 (11:00-11:50 AM) 11 (1:15-2:05 PM)	Man Chun Li 12 (1:15-2:05 PM) 08 (11:00-11:50 AM)	Simon Rubinstein-Salzedo 17 (1:15-2:05 PM) 21 (11:00-11:50 AM)
Aaron Smith 09 (11:00-11:50 AM) 20 (10:00-10:50 AM)	Nikola Penev 14 (1:15-2:05 PM) 24 (2:15-3:05 PM)	Eric Malm 15 (11:00-11:50 AM) 23 (1:15-2:05 PM)	Yu-jong Tzeng 51A

Signature: ______________________________

Instructions: Print your name and student ID number, print your section number and TA's name, write your signature to indicate that you accept the honor code. During the test, you may not use notes, books, calculators. Read each question carefully, and show all your work.
There are ten problems on the pages numbered from 1 to 13, with the total of 140 points. Point values are given in parentheses. You have 3 hours (until 10PM) to answer all the questions.

In the exam all vectors are columns, but sometimes we use transpose to write them horizontally.

Thus $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = [v_1, v_2, \ldots, v_k]^T$.

Similarly $\mathbf{v}^T$ is a row $[v_1, v_2, \ldots, v_k]$.

The dot product of two vectors is denoted as $\mathbf{v} \cdot \mathbf{w}$.

Problem 1. For what values of parameters a and b does the system

$$\begin{cases} x + 2y + 3z = 1 \\ 2x + 4y + a \cdot z = b \end{cases}$$

a) (3 pts.) Has more than one solution?

b) (3 pts.) Has unique solution?

c) (3 pts.) Has no solution?

Problem 2. (6 pts.) Find $\det A$ if

$$A^{-1}B = \begin{bmatrix} 1 & 12 & 34 \\ 0 & 6 & 13 \\ 0 & 0 & 23 \end{bmatrix}$$

and $\det B = 23$.

Problem 3. a) (6 pts.) Write an equation of the plane in $\mathbb{R}^3$ that is passing through the points $(-1, 1, 0)$, $(2, -3, 1)$ and $(2, 3, -2)$.

b) (6 pts.) Let $\mathbf{T}$ be the linear transformation given by multiplication by

$$A = \begin{bmatrix} 3 & 4 & 17 \\ 0 & -3 & 23 \\ 0 & 0 & -1 \end{bmatrix}$$

and let R be a triangle in $\mathbb{R}^3$ whose area is 3. Find the area of the region $\mathbf{T}(R)$.

Problem 4. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of a linear subspace V of $\mathbb{R}^n$.
a) (6 pts.) Show that if $\mathbf{x} \cdot \mathbf{v}_i = 0$ for each $i = 1, 2, 3$, then $\mathbf{x} \cdot \mathbf{v} = 0$ for any $\mathbf{v} \in V$.

b) (6 pts.) Let $V^\perp$ be the set of vectors orthogonal to all the vectors of V, i.e.:

$$V^\perp = \{\mathbf{x} \in \mathbb{R}^n \ ; \ \mathbf{x} \cdot \mathbf{v} = 0 \text{ for all } v \in V\}.$$

Find a matrix A such that $N(A) = V^\perp$. (Your answer should use the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.)

c) (6 pts.) Show that the only vector belonging simultaneously to V and $V^\perp$ is the zero vector. (Hint: consider $\mathbf{v} \cdot \mathbf{v}$ for such a vector $\mathbf{v}$).

Problem 5. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ and $L = \text{span}(\mathbf{v})$. Let P be the orthogonal projection on the line L in $\mathbb{R}^3$.

a) (8 pts.) If $S = \{\mathbf{x} \in \mathbb{R}^3; \mathbf{x} \cdot \mathbf{v} = 0\}$, show that S is a subspace of $\mathbb{R}^3$. Check all three conditions of the linear subspace.

b) (8 pts.) Find a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of S.

c) (4 pts.) Given that $\mathcal{B} = \{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $\mathbb{R}^3$ ($\mathbf{v}_1, \mathbf{v}_2$ are the vectors you found above), find the matrix of P in $\mathcal{B}$?

Problem 6. a) (8 pts.) Find the points on the sphere $x^2 + y^2 + z^2 = 24$ where $f(x, y, z) = 2x + y - z$ has its minimum and maximum values.

b) (3 pts.) What is the geometric meaning of the minimum and maximum points?

Problem 7. a) (9 pts.) Find and classify the critical points of the function

$$f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$$

b) (4 pts.) Is it possible that $f(x, y)$ has a global minimum or maximum?

c) (5 pts.) Find an equation of the tangent plane to the graph of $f(x, y)$ at the point $(2, 2, 8)$. (you do not need to verify that $f(2, 2) = 8$.)

Problem 8. Let A be the matrix

$$A = \begin{bmatrix} 7 & 5 & -7 \\ -5 & -3 & 6 \\ 1 & 1 & 0 \end{bmatrix}$$

a)(5 pts.) Find eigenvectors corresponding to the eigenvalue $\lambda = 1$.

b) (5 pts.) Given that $\mathbf{v} = [-1, 1, 0]$ is an eigenvector of $A = \begin{bmatrix} 7 & 5 & -7 \\ -5 & -3 & 6 \\ 1 & 1 & 0 \end{bmatrix}$, find the corresponding eigenvalue.

c) (5 pts.) Given that the characteristic polynomial of the matrix A is

$$(\lambda - 2)(\lambda - 1)^2$$

determine if the matrix A is diagonalizable. **Justify** your answer.

d) (4 pts.) Using the same characteristic polynomial $(\lambda-2)(\lambda-1)^2$, determine if the matrix A is invertible. **Justify** your answer.

Problem 9. Let f be the following function on $\mathbb{R}^3$:

$$f(x,y,z) = (x+y^2, y+z^2, z+x^2)$$

and let $g(x,y,z) = \exp(x+y+z)$.

a) (5 pts.) Show that the matrix of the derivative of f is:

$$D = \begin{bmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{bmatrix}.$$

b) (5 pts.) Starting from the point $(0, 0, 0) \in \mathbb{R}^3$ in which direction shall one move in order to increase $g(x, y, z)$ fastest?

c) (7 pts.) Calculate the derivative of $g \circ f$.

Problem 10. Let A be the following matrix:

$$A = \begin{bmatrix} 1 & 2 & y \\ 2 & x & 1 \\ 1 & 2 & 2y \end{bmatrix}.$$

a) (4 pts.) Calculate $\det(A)$.

b) (6 pts.) Is the determinant more sensitive to changes in x or y near $x = 1$ and $y = 0$?

Problem	Score	Maximum
1		9
2		6
3		12
4		18
5		20
6		11
7		18
8abc		15
8d		4
9a		5
9bc		12
10		10
Total		140