Math 51 — Spring 2009 — Exam II Solutions

Problem 1. (10 pts.) Let $\mathbf{L}$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^k$. Each of the statements about $\mathbf{L}$ below is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). For each part, decide which, and justify your answer completely.

Each part is 5 points: 2 for a correct answer, 3 more for correct justification.

Note: if the answer is "MAYBE," complete justification must include two examples — one where the statement holds true, and another where it is false. If the answer is "TRUE" or "FALSE," a proper justification cannot be done by giving a single example — only by giving a proof.

a) If vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent, then the vectors $\mathbf{L}(\mathbf{v}_1), \mathbf{L}(\mathbf{v}_2), \ldots, \mathbf{L}(\mathbf{v}_m)$ are linearly independent as well. T F **[MAYBE]**

As noted above, we must give two examples: one where this statement holds, and one where it does not. There are many possibilities for each.

- Suppose $n = k$ and $\mathbf{L} = \mathbf{Id_n}$, the identity transformation; then $\mathbf{L}(\mathbf{v}_1) = \mathbf{v}_1$, $\mathbf{L}(\mathbf{v}_2) = \mathbf{v}_2$, ..., $\mathbf{L}(\mathbf{v}_m) = \mathbf{v}_m$ for any choice of vectors. So if we assume $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent, then certainly $\mathbf{L}(\mathbf{v}_1), \mathbf{L}(\mathbf{v}_2), \ldots, \mathbf{L}(\mathbf{v}_m)$ are independent as well.
- If we take $\mathbf{L}$ to be the zero transformation ($\mathbf{L}(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$), then $\mathbf{L}(\mathbf{v}_1) = \cdots = \mathbf{L}(\mathbf{v}_m) = \mathbf{0}$, and so in this case $\mathbf{L}(\mathbf{v}_1), \ldots, \mathbf{L}(\mathbf{v}_m)$ are linearly dependent.

b) If vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent, then the vectors $\mathbf{L}(\mathbf{v}_1), \mathbf{L}(\mathbf{v}_2), \ldots, \mathbf{L}(\mathbf{v}_m)$ are linearly dependent as well. **[T]** F MAYBE

Since $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly dependent, we may find scalars $c_1, \ldots, c_m$, not all equal to 0, such that $c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$. But then $\mathbf{0} = \mathbf{L}(\mathbf{0}) = c_1\mathbf{L}(\mathbf{v}_1) + \cdots + c_m\mathbf{L}(\mathbf{v}_m)$, since $\mathbf{L}$ is a linear transformation, and so $\mathbf{L}(\mathbf{v}_1), \ldots, \mathbf{L}(\mathbf{v}_m)$ are linearly dependent (because not all c_i are equal to 0).

Problem 2. (12 pts.)

a) Complete the following sentence to make a true statement: A linear transformation $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if

(4 points) Choose your favorite one; they're all equivalent (but be careful to note the distinction between the function $\mathbf{L}$ and its matrix — these are different objects):

- ... $\mathbf{L}$ is both one-to-one *and* onto.
- ... $n = m$ and $\mathbf{L}$ is onto.
- ... $n = m$ and $\mathbf{L}$ is one-to-one.
- ... the matrix of $\mathbf{L}$ with respect to the standard basis (that is, the matrix A such that $\mathbf{L}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$) has reduced row echelon form equal to I_n.
- ... the matrix of $\mathbf{L}$ is square and has rank n.
- ... the matrix of $\mathbf{L}$ is square and has linearly independent columns.
- ... the matrix of $\mathbf{L}$ is square and has linearly independent rows.
- ... the matrix of $\mathbf{L}$ is square and has nonzero determinant.
- and so on...

For parts (b) and (c), suppose

- $\mathbf{S}$ is the linear transformation given by multiplication by the matrix $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$, and
- $\mathbf{T}$ is the linear transformation given by multiplication by the matrix $B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 2 \end{bmatrix}$.

b) Is it possible to define the composition $\mathbf{S} \circ \mathbf{T}$? If so, is it invertible? Explain completely.

(4 points) $\mathbf{S} \circ \mathbf{T}$ is defined, since an output of $\mathbf{T}$ has the same number of components (two) as an input of $\mathbf{S}$. However, it is not invertible, which we can see in two different ways.

Here is an elegant solution: note that $\mathbf{T}$, being a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$, cannot be one-to-one, since B is a matrix that must have a nontrivial null space (it has more columns than rows). Therefore the composition $\mathbf{S} \circ \mathbf{T}$ cannot be one-to-one, so it is not invertible.

Explicit solution: the matrix of $\mathbf{S} \circ \mathbf{T}$ is $AB = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -5 & -4 \\ 3 & 9 & 6 \\ 1 & 3 & 2 \end{bmatrix}$; it's not invertible since the second row is a multiple of the third (so its determinant is 0).

c) Is it possible to define the composition $\mathbf{T} \circ \mathbf{S}$? If so, is it invertible? Explain completely.

(4 points) $\mathbf{T} \circ \mathbf{S}$ is defined, since an output of $\mathbf{S}$ has the same number of components (three) as an input of $\mathbf{T}$. The matrix of $\mathbf{T} \circ \mathbf{S}$ is the product

$$BA = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 2 & 10 \end{bmatrix}.$$

This matrix has non-zero determinant and is therefore invertible; thus, $\mathbf{T} \circ \mathbf{S}$ is invertible.

Problem 3. (10 pts.) Find the inverse of the matrix

$$\begin{bmatrix} 5 & 0 & -2 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

We form an augmented matrix with the identity and use row reduction:

$$\left[\begin{array}{ccc|ccc} 5 & 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$
$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} r_1' = r_1 + 2r_3 \\ r_2' = r_2 \\ r_3' = r_3 \end{array}$$
$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & 2 & 0 & 5 \end{array}\right] \begin{array}{l} r_1'' = r_1' \\ r_2'' = r_2' - r_1' \\ r_3'' = r_3' + 2r_1' \end{array} .$$

Therefore

$$\begin{bmatrix} 5 & 0 & -2 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 2 & 0 & 5 \end{bmatrix}.$$

Problem 4. (12 pts.) Let A be an $n \times k$ matrix, where $k \leq n$, whose columns are mutually orthogonal unit vectors in $\mathbb{R}^n$; that is,

$$A = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_k \\ | & & | \end{bmatrix}, \text{ where } \begin{cases} \mathbf{v}_i \cdot \mathbf{v}_j = 0 \text{ for all } i \neq j, \text{ and} \\ \|\mathbf{v}_i\| = 1 \text{ for all } i \end{cases}$$

a) Show that the product $A^T A$ is the identity matrix, and find the size of that identity matrix.

(3 points)

$$A^T A = \begin{bmatrix} — \mathbf{v}_1^T — \\ — \mathbf{v}_2^T — \\ \vdots \\ — \mathbf{v}_k^T — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_1 \cdot \mathbf{v}_k \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \ldots & \mathbf{v}_2 \cdot \mathbf{v}_k \\ & \vdots & & \\ \mathbf{v}_k \cdot \mathbf{v}_1 & \mathbf{v}_k \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_k \cdot \mathbf{v}_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_k$$

The size of the identity matrix is $k \times k$.

b) Let $\mathbf{b} = \begin{bmatrix} 1 & 2 & \cdots & n \end{bmatrix}^T$. Use the matrices A, A^T, and the vector $\mathbf{b}$ to write a solution to the system $A\mathbf{x} = \mathbf{b}$, or show that the system does not have solutions.

(3 points) We first note that both the matrix A and the vector $\mathbf{b}$ have n rows; thus, it makes sense to define the linear system $A\mathbf{x} = \mathbf{b}$, where the unknowns vector $\mathbf{x}$ is taken to have k components.

There can be only one solution to this system; to see this, we multiply both sides of $A\mathbf{x} = \mathbf{b}$ by A^T to obtain $A^T A\mathbf{x} = A^T\mathbf{b}$. But by part (a), $A^T A\mathbf{x} = I_k\mathbf{x} = \mathbf{x}$, so $\boxed{\mathbf{x} = A^T\mathbf{b}}$.

As seen in Levandosky Chapter 9, such a solution exists if and only if $\mathbf{b}$ lies in the column space of A.

Notes: The most common misconception was to draw a connection between the existence of a solution and the question of A's invertibility, and thus also to the size of the matrix A. But note that A need not be a square matrix for there to be a solution to this system; thus, the non-existence of A^{-1} does not preclude a solution.

c) Give an example of such a matrix A (i.e., whose columns are mutually orthogonal unit vectors), but for which the product AA^T is *not* the identity matrix.

(3 points) Many answers are possible. For example, let $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then since A has only one column vector and it's a unit vector, it certainly satisfies the description required. We have that

$$AA^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is not the identity matrix.

Notes: It's impossible to give such an example if $k = n$. So we should consider a nonsquare matrix with more rows than columns. (And please compute AA^T to verify your answer; don't just write a matrix and let the grader verify its value for you.)

d) Suppose $k < n$; explain why the formula $\det(A^T A) = \det(A^T)\det(A)$ fails for such a matrix A.

(3 points) Since $k < n$, A and A^T are not square matrices; thus, their determinants are not defined. Therefore the equation does not hold because the right hand side is undefined.

Notes: To simply say "$\det(A^T A) = \det(A^T)\det(A)$ fails because A and A^T are not square matrices" didn't receive any credit, because it's already clear from $k < n$ that they are not square matrices. The crux is *explaining* why the equation does not hold in this case.

Furthermore, the fact that A and A^T are not square matrices does *not* mean their determinants are zero. When one says that a determinant is zero, that implies the determinant *exists and is equal to zero.* In the case of a non-square matrix, all one can say is something like "cannot find deteterminant" or "does not exist."

Problem 5. (12 pts.) Let $\mathbf{v}_1 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$, and $V = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{v}_1 = 0\}$ be the set of vectors orthogonal to $\mathbf{v}_1$. It is a fact (which you do not have to prove) that V has a basis $\{\mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_2 = \begin{bmatrix}1 & 0 & -1\end{bmatrix}^T \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix}0 & 1 & -1\end{bmatrix}^T$$

a) Show that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$.

(4 points) Three vectors form a basis for $\mathbb{R}^3$ if and only if they are linearly independent, which we check by forming a matrix C with columns $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ and row-reducing it:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $\operatorname{rref}(C) = I_3$ the vectors form a basis. Alternatively, we could have calculated that $\det(C) = 3 \neq 0$, which also shows that $\mathcal{B}$ is a basis.

Notes: Several claimed *without proof* that since $\{\mathbf{v}_2, \mathbf{v}_3\}$ is a basis for V and $\mathbf{v}_1$ is perpendicular to $\mathbf{v}_2$ and $\mathbf{v}_3$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent. The assertion is true, but since this is basically just restating the given information from the question, it received zero points in most cases. There is no need to rely on unproven claims when you have the vectors themselves available for a simple calculation.

b) Let $\mathbf{P} : \mathbb{R}^3 \to \mathbb{R}^3$ denote projection onto the line $L = \operatorname{span}(\mathbf{v}_1)$. What is the matrix of $\mathbf{P}$ with respect to $\mathcal{B}$? (Find its entries explicitly.)

(4 points) Since $\mathbf{v}_1$ is perpendicular to $\mathbf{v}_2$ and $\mathbf{v}_3$,

$$\begin{aligned} \mathbf{P}(\mathbf{v}_1) &= \mathbf{v}_1 \\ \mathbf{P}(\mathbf{v}_2) &= \mathbf{0} \\ \mathbf{P}(\mathbf{v}_3) &= \mathbf{0} \end{aligned}$$

Therefore,

$$B = [\mathbf{P}]_{\mathcal{B}} = \begin{bmatrix} | & | & | \\ [\mathbf{P}(\mathbf{v}_1)]_{\mathcal{B}} & [\mathbf{P}(\mathbf{v}_2)]_{\mathcal{B}} & [\mathbf{P}(\mathbf{v}_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ [\mathbf{v}_1]_{\mathcal{B}} & [\mathbf{0}]_{\mathcal{B}} & [\mathbf{0}]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c) What is the matrix of $\mathbf{P}$ with respect to the standard basis of $\mathbb{R}^3$? (You may leave your answer as a product of matrices and their inverses, without calculating, but you must explicitly write the entries of the matrices that make up the parts of your product(s) or inverse(s).)

(4 points) There are two ways to solve this part. The first is by the formula for orthogonal projection onto a line; make a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}_1$:

$$\mathbf{u} = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Then $\mathbf{P}(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{u})\mathbf{u}$, so

$$A = \begin{bmatrix} | & | & | \\ \mathbf{P}(\mathbf{e}_1) & \mathbf{P}(\mathbf{e}_2) & \mathbf{P}(\mathbf{e}_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ (\mathbf{e}_1 \cdot \mathbf{u})\mathbf{u} & (\mathbf{e}_2 \cdot \mathbf{u})\mathbf{u} & (\mathbf{e}_3 \cdot \mathbf{u})\mathbf{u} \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | \\ u_1\mathbf{u} & u_2\mathbf{u} & u_3\mathbf{u} \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_1u_2 & u_2^2 & u_2u_3 \\ u_1u_3 & u_2u_3 & u_3^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

The second way is by using part (b) and a change of basis:

$$A = CBC^{-1} = C[P]_{\mathcal{B}}C^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}^{-1}$$

Notes: As the problem statement said, leaving your answer in this form was sufficient for full credit, but if you calculated C^{-1} incorrectly and used it in your answer, you lost a point.

Problem 6. (10 pts.)

a) Complete the following sentence: λ is defined to be an eigenvalue of the matrix A if

(5 points) Choose any of the following; they're all equivalent (as long as we assume A is $n \times n$):

- ... $\det(\lambda I_n - A) = 0$.
- ... $\text{rank}(\lambda I_n - A) < n$.
- ... there exists a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\lambda \mathbf{v} = A\mathbf{v}$.

Notes: The most common mistake in answers to this question was to give a condition such as $A\mathbf{v} = \lambda \mathbf{v}$ without stipulating that $\mathbf{v} \neq \mathbf{0}$. This distinction is crucial, since *every* scalar λ is a solution to $A\mathbf{0} = \lambda \mathbf{0}$.

b) Let

$$A = \begin{bmatrix} * & * & * \\ 1 & 7 & -1 \\ * & * & * \end{bmatrix}$$

where $*$ denotes an entry of the matrix A that is not known to us. Assuming that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigenvector of the matrix A, find the corresponding eigenvalue.

(5 points) We know the equation

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} * & * & * \\ 1 & 7 & -1 \\ * & * & * \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} * \\ 12 \\ * \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

must hold. It follows that $12 = 2\lambda$, by considering the middle entry. Consequently $\boxed{\lambda = 6}$.

Problem 7. (12 pts.) Let A be the following matrix, where a is an arbitrary real number:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & a & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

It is a fact that the eigenvalues of A are always $1, a, 2$ (you don't have to verify this!).

a) Suppose a is not equal to 1 or 2; explain why the matrix A is diagonalizable. (*Hint:* it's possible to avoid calculations.)

(4 points) Recall that an $n \times n$ matrix A is diagonalizable precisely when there exists a basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$ where each $\mathbf{v}_i$ is an eigenvector of A. Such a basis is termed an *eigenbasis.*

Proposition 23.3 of Levandosky asserts that if an $n \times n$ matrix has n distinct eigenvalues, then for eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ corresponding to these eigenvalues, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent and consequently an eigenbasis.

In this case, if $1, a, 2$ are distinct, this principle applies, and A is diagonalizable.

b) Suppose $a = 1$; show that A is diagonalizable. (*Hint:* try to find an eigenbasis for A.)

(4 points) 2 is an eigenvalue of A, and we can find a corresponding eigenvector $\mathbf{v}$ as a nonzero vector satisfying[a]

$$(2 \cdot I_3 - A)\mathbf{v} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{v} = \mathbf{0}.$$

We take $\mathbf{v} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T$.

We consider the set of eigenvectors of A associated with the eigenvalue 1. These are exactly the nonzero vectors $\mathbf{w}$ satisfying

$$(1 \cdot I_3 - A)\mathbf{w} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{w} = \mathbf{0}.$$

The set of solutions to this equation is

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Taking

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

we observe that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}\}$ is linearly independent and consequently an eigenbasis. Since an eigenbasis exists, the matrix A is diagonalizable.

[a] This and all further eigenvector calculations amount to solving a system of linear equations. Failure to solve such a system correctly was by far the most common error in answers to this question.

For easy reference, $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & a & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

c) Suppose $a = 2$; show that A is *not* diagonalizable. (Same hint as part (b).)

(4 points) 1 is an eigenvalue of the matrix A; the corresponding eigenvectors are the nonzero vectors satisfying

$$(1 \cdot I_3 - A)\mathbf{v} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{v} = \mathbf{0}.$$

The vectors satisfying this equation are precisely those $\mathbf{v}$ in $\text{span}(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T)$. Since $\text{span}(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T)$ is 1-dimensional, any two eigenvectors corresponding to the eigenvalue 1 are linearly dependent on each other.

2 is an eigenvalue of the matrix A; the corresponding eigenvectors are the nonzero vectors satisfying

$$(2 \cdot I_3 - A)\mathbf{w} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{w} = \mathbf{0}.$$

The vectors satisfying this equation are precisely those $\mathbf{w}$ in $\text{span}(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T)$. Since $\text{span}(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T)$ is 1-dimensional, any two eigenvectors corresponding to the eigenvalue 2 are linearly dependent on each other.

If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a set of eigenvectors of A, then at least two of the $\mathbf{u}_i$ must correspond to the same eigenvalue, be it 1 or 2. These two vectors must be linearly dependent on each other, from which it follows that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, and therefore not a basis of $\mathbb{R}^3$. Since A does not allow us to find an eigenbasis, A is not diagonalizable.

Notes: Failure to compute the eigenvectors correctly was the most common error. Another common family of errors stemmed from a misunderstanding about the number of different eigenvectors a matrix admits. A matrix for which we can find an eigenvector[a] $\mathbf{v}$ admits infinitely many eigenvectors since any nonzero multiple of $\mathbf{v}$ is an eigenvector. One way or another, this question is concerned with finding the biggest possible set of *linearly independent* eigenvectors, which is a finite set.

[a]If we know complex numbers, this is any matrix; if not, it is any matrix with an eigenvalue, which includes all matrices in this question.

Problem 8. (12 pts.) Let $Q(x, y, z) = x^2 + z^2 + 2xy + 6yz$ be a quadratic form on $\mathbb{R}^3$.

a) Write the symmetric matrix A associated to the quadratic form Q.

(3 points)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

b) Determine the definiteness of Q; use any method you like, but justify your answer.

(3 points) The easiest approach is to find two different points so that the values of Q at those points differ in signs. For example:

$$Q(1, 0, 0) = 1 > 0$$
$$Q(1, -1, 0) = -7 < 0.$$

Hence Q is $\boxed{\text{indefinite}}$.

Note: Some solutions just computed the determinant of A, which is -10, and claimed that since $\det A$ is negative, Q must be indefinite. This is always true for a 2×2 matrix, but for a 3×3 matrix this is not necessarily correct. (A 3×3 matrix corresponding to a negative definite Q would have three negative eigenvalues and thus have a negative determinant.)

But there's a way around this obstacle in this case: you can make use of the additional fact that $\operatorname{tr}(A)$ is positive, and argue that (i) the eigenvalues of A cannot all be negative (for otherwise $\operatorname{tr}(A)$ would be negative); while on the other hand, (ii) the eigenvalues cannot all be positive (for otherwise $\det(A)$ would be positive). With this extended argument, you'd then be able to conclude that Q must be indefinite.

For easy reference, $Q(x, y, z) = x^2 + z^2 + 2xy + 6yz$.

c) Let $\mathbf{a} = (1, 2, 1)$. Find $DQ(\mathbf{a})$, the matrix of partial derivatives of Q evaluated at $\mathbf{a}$.

(3 points)

$$\begin{aligned} DQ(\mathbf{a}) &= \begin{bmatrix} 2x + 2y & 2x + 6z & 2z + 6y \end{bmatrix} \Big|_{(1,2,1)} \\ &= \begin{bmatrix} 6 & 8 & 14 \end{bmatrix} \end{aligned}$$

Notes: A common mistake was to give the transpose of $DQ(\mathbf{a})$ instead (i.e a 3×1 matrix). Another common mistake was forgetting to evaluate the partial derivatives at the given point $\mathbf{a}$.

d) Evaluate $\dfrac{\partial^2 Q}{\partial x^2}$ and $\dfrac{\partial^2 Q}{\partial y \partial z}$ at the point $\mathbf{a} = (1, 2, 1)$.

(3 points) These are second-order partial derivatives, so we use the calculation in part (c) above:

$$\frac{\partial^2 Q}{\partial x^2}(1, 2, 1) = \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial x}\right)\Big|_{(1,2,1)} = \frac{\partial}{\partial x}(2x + 2y)\Big|_{(1,2,1)} = 2$$

$$\frac{\partial^2 Q}{\partial y \partial z}(1, 2, 1) = \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial z}\right)\Big|_{(1,2,1)} = \frac{\partial}{\partial y}(2z + 6y)\Big|_{(1,2,1)} = 6$$

Problem 9. (10 pts.) Let $f(x, y) = x^3 + xy + 1$.

a) Find an equation that defines the level set of f containing the point $(x, y) = (1, -1)$. On the set of axes below, sketch this level set. (Be sure to label your axes.)

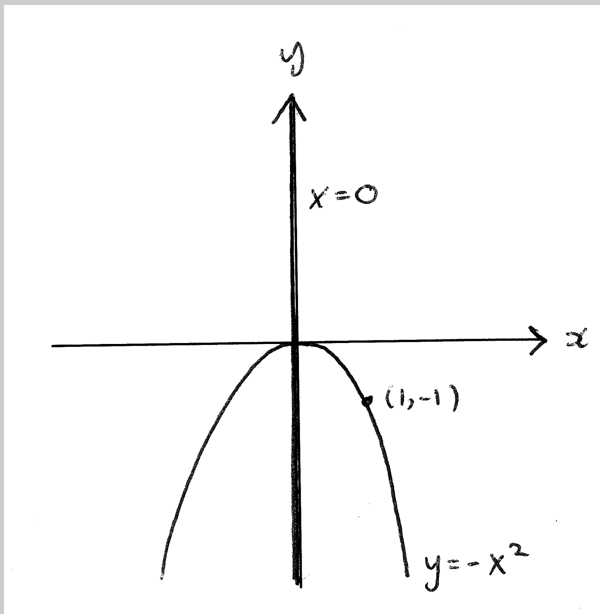
(5 points) At the point $(1, -1)$, $f(1, -1) = 1 - 1 + 1 = 1$. So the level set of f at level 1 contains the point $(1, -1)$. The defining equation is

$$f(x, y) = x^3 + xy + 1 = 1,$$

which implies

$$0 = x^3 + xy = x(x^2 + y), \qquad \text{so that} \qquad x = 0 \quad \text{or} \quad x^2 + y = 0.$$

Hence the level set containing $(1, -1)$ consists of the line $x = 0$ and the parabola $y = -x^2$.



Notes: Many responses missed the case $x = 0$. They often claimed that $y = -x^3/x = -x^2$, and failed to realize that this is true only when $x \neq 0$.

b) Find an equation for the plane in $\mathbb{R}^3$ tangent to the graph of $z = f(x, y)$ at the point $(0, 0, 1)$.

(5 points) The equation for the required plane is given by

$$z = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0),$$

where $f(0, 0) = 1$ since we are told that the graph of $z = f(x, y)$ contains the point $(0, 0, 1)$, and

$$f_x(0, 0) = (3x^2 + y)\Big|_{(0,0)} = 0 \qquad \text{and} \qquad f_y(0, 0) = x\Big|_{(0,0)} = 0.$$

Hence the equation for the plane is $\boxed{z = 1}$.