# Math 51- Winter 2008 - Midterm Exam I

Please circle the name of your TA:

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|---|---|---|---|
| Zachary Cohn | José Perea | Nikola Penev | Man Chun Li |
| Daniel Mathews | Theodora Bourni | Anssi Lahtinen | Isidora Milin |

Circle the time your TTh **section** meets: 10:00 11:00 1:15 2:15

Your name (print): SOLUTIONS

Student ID:

Sign to indicate that you accept the honor code:

**Instructions:** Circle your TA's name and the time that you attend the TTh section. Read each question carefully, and show all your work. You have 90 minutes to do all the problems. During the test, **you may NOT use any notes, books, or calculators**.

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Maximum | 12 | 12 | 12 | 15 | 12 | 15 | 12 | 10 | 100 |
| Score | | | | | | | | | |

**Formulas you may use:**

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

**Problem 1.** (12 points total)

(a) Write the equation of the line passing through the point $(2,1)$ and with normal vector $\mathbf{n} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

eq $\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 2 \\ x_2 - 1 \end{pmatrix} = 0$$

$$-1(x_1 - 2) + x_2 - 1 = 0$$

$$-x_1 + x_2 = -1.$$

(b) What is the parametric equation of this line?

$$\begin{cases} x_1 = 1 + t \\ x_2 = t \end{cases}$$ $t$ real param (free variable)

**Problem 2.** (12 points) Given two vectors $u$ and $v$ such that $\|u\| = \|v\|$ show that $u + v$ and $u - v$ are orthogonal (perpendicular). (Hint: Use dot product)

$$(u+v)\cdot(u-v) = u \cdot u - \cancel{u \cdot v} + \cancel{u \cdot v} - v \cdot v$$

$$= \|u\|^2 - \|v\|^2 = 0 \quad \text{since } \|u\| = \|v\|$$

So $u+v$ and $u-v$ are othogonal

**Problem 3.** (12 points) Consider system of equations

$$\begin{cases} x + y = 2 \\ x + ay = b. \end{cases}$$

where $a$ and $b$ are constants, and $x$ and $y$ are the unknowns. Determine all the values of $a$ and $b$ for which the system above has:

(a) no solution $a = 1$ but $b \neq 2$

(b) a unique solution $a \neq 1$

(c) exactly two solution never

(d) more than two solutions $a = 1, b = 2$

**Explain your answer below!**

algebraically: $\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & a & b \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & a-1 & b-2 \end{array}\right)$

(a) no soln if $a-1=0$ but $b-2 \neq 0$
(get inconsistent)

(b) unique soln if $a \neq 1$ (no free var).

(c) never, a lin system can never have exactly 2 soln

(d) when $a=1$, $b=2$ get infinitely many soln (free var).

geometrically: We are looking at the intersection of 2 lines

(a) no soln if lines are parallel and not equal so when $a=1$ but $b \neq 2$

(b) unique soln if lines not parallel so $a \neq 1$

(c) never, two lines never intersect in exactly 2 pts

(d) when same line so $a=1$, $b=2$

**Problem 4.** (15 points total) Let $\{u, v, w\}$ be three linearly independent vectors.

(a) (5 points) Show that $\{u+v, u-v\}$ are linearly independent.

assume $x_1(u+v) + x_2(u-v) = 0$ (Solve for $x_1, x_2$)

so $(x_1+x_2)u + (x_1-x_2)v = 0$

but $u, v$ lin indep so $x_1+x_2 = 0$, $x_1 - x_2 = 0$

and so $x_1 = 0$, $x_2 = 0$ so

$u+v$, $u-v$ are lin indep.

(b) (5 points) Are $\{u, u+v, u-v\}$ linearly independent? Please explain your answer.

No, for example

$-2u + 1(u+v) + 1(u-v) = 0$

(c) (5 points) Is $w$ in the span of $\{u+w, v+w\}$? Please explain your answer.

No. If we assume that $w$ is in the span,

$w = x_1(u+w) + x_2(v+w)$ get

$x_1 u + x_2 v + (x_1+x_2-1)w = 0$ but $u, v, w$ lin indep

so $x_1 = 0$, $x_2 = 0$, $x_1 + x_2 - 1 = 0$ no soln!

so $w$ cannot be in the span above.

**Problem 5.** (15 points total) Consider the matrix $A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

(a) find a basis for $N(A)$

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$x_2, x_4$ free var, $x_1, x_3$ pivot var $\quad x_1 = -x_2 - x_4$

$x_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_2 - x_4 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

so $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $N(A)$

(b) find a basis for $C(A)$

basis for $C(A)$ is obtained from columns of A corresponding to pivot var so

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ form a basis for $C(A)$

(c) given that $A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, find all solutions to the equation $Ax = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$x = x_p + x_h = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$x_2, x_4$ free var. (param)

**Problem 6.** (12 points total) Which of the following are linear subspaces of $\mathbb{R}^2$? Please explain your answer.

(a) the set $V = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0\}$

Not a linear subspace

For example $x = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is in $V$ but

$(-1) \cdot x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not in $V$

(b) the set $V = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$

yes, it is a subspace

For example it is the null space of $A = (1, 0)$

(could also check by hand the properties)

**Problem 7.** (12 pts total) Consider the following linear subspace of $\mathbb{R}^3$:

$$V = \{(x_1, x_2, x_3) \mid x_1 - x_2 + x_3 = 0\}.$$

(a) find a basis for $V$

$x_2, x_3$ free var $\quad x_1 = x_2 - x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

so $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $V$

(b) give an example of a matrix $A$ such that $N(A) = V$.

$$A = (1, -1, 1)$$

(c) give an example of a matrix $A$ such that $C(A) = V$.

$$A = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**Problem 8.** (10 points) Circle T or F to mark each of the following true or false. Explanations are not required for this problem.

(a) the dot product of two vectors in $\mathbb{R}^3$ is a vector in $\mathbb{R}^3$ **T** **F**

No, it is a scalar

(f) any three vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$. **T** **F**

No, all vectors could be equal

(b) a system of 3 linear equations with 5 unknowns cannot have a unique solution. **T** **F**

yes, we must have at least 2 free variables
so cannot have unique soln

(c) a system of 5 linear equations with 3 unknowns cannot have more then one solution. **T** **F**

yes it could, all 5 eq could be the same one

(d) for all matrices $A$, the column space of $A$ equals the column space of the rref (A) **T** **F**

no, see Problem 5(b)

(e) for all matrices $A$, the null space of $A$ equals the null space of the rref (A). **T** **F**

yes, the soln of $Ax = 0$ is preserved
by row operations.

(g) if $A$ is a $4 \times 2$ matrix then $\dim N(A) \leq 2$ **T** **F**

$\dim N(A) = \#$ free var $\leq 2 =$ total $\#$ of var

(h) if $A$ is a $2 \times 4$ matrix then $\dim N(A) \geq 2$ **T** **F**

there are at least 2 free variables

(i) there are $3 \times 6$ matrices with $\dim N(A) = 3$ and $\dim C(A) = 3$. **T** **F**

yes, eg $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$ (3 free var and 3 pivot var)

(j) there are $6 \times 3$ matrices with $\dim N(A) = 3$ and $\dim C(A) = 3$. **T** **F**

No, there are 3 variables, so there is no
room for 3 free variables and 3 pivot var