Math 51 Exam 2 Solutions — May 20, 2008

1. (8 points) Compute the following determinant:

$$\begin{vmatrix} 2 & 4 & -2 & -1 \\ 1 & 1 & -1 & 0 \\ 3 & 2 & -1 & -1 \\ 2 & 2 & -3 & 1 \end{vmatrix}$$

Due to the lack of many zeros, it's convenient to begin with a few row operations:

$$\begin{aligned}
\begin{vmatrix} 2 & 4 & -2 & -1 \\ 1 & 1 & -1 & 0 \\ 3 & 2 & -1 & -1 \\ 2 & 2 & -3 & 1 \end{vmatrix} \begin{matrix} \textit{swap} \\ \textit{swap} \\ \\ \\ \end{matrix}
&= - \begin{vmatrix} 1 & 1 & -1 & 0 \\ 2 & 4 & -2 & -1 \\ 3 & 2 & -1 & -1 \\ 2 & 2 & -3 & 1 \end{vmatrix} \begin{matrix} \\ -2I \\ -3I \\ -2I \end{matrix} \\
&= - \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{vmatrix} \begin{matrix} \\ +2III \\ \\ \\ \end{matrix} \\
&= - \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{vmatrix} \begin{matrix} \\ \\ \cdot(-1) \\ \\ \end{matrix} \\
&= \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix} \qquad \text{[expand along first column]} \\
&= (1) \begin{vmatrix} 0 & 4 & -3 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{vmatrix} \qquad \text{[expand along first column]} \\
&= (1) \left(-(1) \begin{vmatrix} 4 & -3 \\ -1 & 1 \end{vmatrix} \right) \\
&= -\left(4 \cdot 1 - (-3)(-1)\right) \\
&= \boxed{-1}
\end{aligned}$$

2. (15 points) Let $\mathcal{B}$ be the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

(You don't have to check that $\mathcal{B}$ is a basis.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by the following formulas:

$$T(\mathbf{v}_1) = \mathbf{v}_2, \quad T(\mathbf{v}_2) = \mathbf{v}_3, \quad T(\mathbf{v}_3) = \mathbf{v}_1.$$

(a) Find the matrix of T with respect to the basis $\mathcal{B}$.

(4 points) Write B for the matrix of T with respect to $\mathcal{B}$. Recall that the i^{th} column of B is $[T(\mathbf{v}_i)]_{\mathcal{B}}$, i.e., the *coordinates* of the vector $T(\mathbf{v}_i)$ *with respect to* $\mathcal{B}$. Thus,

$$B = \begin{bmatrix} | & | & | \\ [T(\mathbf{v}_1)]_{\mathcal{B}} & [T(\mathbf{v}_2)]_{\mathcal{B}} & [T(\mathbf{v}_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ [\mathbf{v}_2]_{\mathcal{B}} & [\mathbf{v}_3]_{\mathcal{B}} & [\mathbf{v}_1]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Notes: The most common mistake was to forget to express the outputs of T in *coordinates* with respect to the basis $\mathcal{B}$; thus, many people instead used the standard coordinates of the $\mathbf{v}_i$ vectors as the columns for their matrix.

(b) Find the matrix of T with respect to the standard basis.

(7 points) Note that we seek the matrix A satisfying $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^3$.

Solution 1: The technique most people used was to use the change-of-basis formula

$$A = CBC^{-1},$$

where C is the matrix whose i^{th} column is $\mathbf{v}_i$. This technique is correct, but it is easy to make mistakes when computing the inverse of a matrix, and the operations involved take a lot of time. (Also, some people wrote $A = C^{-1}BC$, which is incorrect.) For completeness, here is a summary of the computations involved: if C is the change-of-basis matrix for $\mathcal{B}$, then

$$C = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ and we must compute (via row reduction) that } C^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

Thus,

$$\begin{aligned} A = CBC^{-1} &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}. \end{aligned}$$

Solution 2: A more efficient approach is to write the standard basis vectors in terms of the vectors in $\mathcal{B}$. If we inspect these vectors, we can see that $3\mathbf{e}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$; applying T to both sides (and using linearity) we get

$$T(3\mathbf{e}_3) = T(\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3) = T(\mathbf{v}_1)+T(\mathbf{v}_2)+T(\mathbf{v}_3) = \mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_1 = 3\mathbf{e}_3, \quad \text{so} \quad T(\mathbf{e}_3) = \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\text{Now,} \qquad \mathbf{e}_1 = \mathbf{e}_3 - \mathbf{v}_2, \quad \text{so} \quad T(\mathbf{e}_1) = T(\mathbf{e}_3 - \mathbf{v}_2) = T(\mathbf{e}_3) - T(\mathbf{v_2}) = \mathbf{e}_3 - \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\text{and} \qquad \mathbf{e}_2 = \mathbf{e}_3 - \mathbf{v}_3, \quad \text{so} \quad T(\mathbf{e}_2) = T(\mathbf{e}_3 - \mathbf{v}_3) = T(\mathbf{e}_3) - T(\mathbf{v}_3) = \mathbf{e}_3 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$$

$$\text{Thus,} \qquad A = \begin{bmatrix} | & | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \\ | & | & | \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

(c) Find a matrix X such that $\quad X \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$

(Hint: is there a relation between X and T^2?)

(4 points) **Solution 1:** If M is any matrix, the i^{th} column of XM is just (X times the i^{th} column of M). Thus, in the product given above, by inspecting each column in turn, we find

$$X\mathbf{v}_1 = \mathbf{v}_3, \quad X\mathbf{v}_2 = \mathbf{v}_1, \quad X\mathbf{v}_3 = \mathbf{v}_2.$$

Since $\mathbf{v}_3 = T(\mathbf{v}_2) = T(T(\mathbf{v}_1)) = T^2(\mathbf{v}_1)$, etc., we conclude that X is the matrix of $T^2 = T \circ T$ *with respect to the standard basis.* On the other hand, we know that the matrix of $T \circ T$ is A^2.

$$\text{Thus,} \qquad X = A^2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Notes: A very common mistake was to write $X = B^2$ — but everything here is written in terms of matrix mulitplication, not in terms of coordinates with respect to $\mathcal{B}$, so everything is in terms of the standard basis. (But in terms of change-of-basis, we do have $X = CB^2C^{-1}$.)

Solution 2: Many people did not find a relation between X and T^2; however, some who solved part (b) using the change-of-basis formula, recognized the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ as the matrix C, whose inverse was already computed. Thus, we can multiply both sides of the given equation on the right by C^{-1}; we find that

$$X = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

3. (16 points) Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

(2 points each)

[✓ **T**] **F** If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is also a basis for $\mathbb{R}^n$, then T must be invertible.

T is onto, since if $\mathbf{w}$ is an arbitrary vector in $\mathbb{R}^n = \operatorname{span}(T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n))$, then there are $c_1, \ldots, c_n$ such that $\mathbf{w} = c_1 T(\mathbf{v_1}) + \cdots + c_n T(\mathbf{v_n}) = T(c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n})$, and thus $\mathbf{w}$ is in the image of T. Thus, the matrix of T is both square and of rank n; it follows that T is invertible.

T [✓ **F**] It is possible to find a 3×2 matrix A and a 2×3 matrix B, satisfying $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Using A and B, we could form $A' = \left[\begin{array}{cc} & 0 \\ A & 0 \\ & 0 \end{array}\right]$ and $B' = \left[\begin{array}{ccc} & B & \\ 0 & 0 & 0 \end{array}\right]$ satisfying $A'B' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; but then A' and B' would be 3×3 matrices having determinant 0, so their product couldn't possibly have determinant 1. Thus, such A and B can't exist.

[✓ **T**] **F** It is possible to find a 2×3 matrix C and a 3×2 matrix D, satisfying $CD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For example, let $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

T [✓ **F**] Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with matrix A, such that whenever R is a region in $\mathbb{R}^2$, then the area of $T(R)$ is equal to the area of R. It follows that $\det(A) = 1$.

The area of $T(R)$ is equal to $|\det(A)|$ times the area of R, so it might be true that $\det(A) = -1$.

T [✓ **F**] If $\mathcal{B}$ is an orthonormal basis for $\mathbb{R}^n$, and C is its associated change-of-basis matrix, then it follows that C is a symmetric matrix.

All that we know for certain is that C is an *orthogonal* matrix, i.e., that $C^T C = I_n$.

[✓ **T**] **F** If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A and B are the matrices of T with respect to two different bases of $\mathbb{R}^n$, then $\det(A) = \det(B)$.

There is some C such that $A = CBC^{-1}$, and so $\det(A) = \det(C)\det(B)\det(C)^{-1} = \det(B)$.

[✓ **T**] **F** If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A and B are the matrices of T with respect to two different bases of $\mathbb{R}^n$, then A and B have the same characteristic polynomial.

There is C such that $A = CBC^{-1}$, and so $\lambda I_n - A = \lambda CC^{-1} - CBC^{-1} = C(\lambda I_n - B)C^{-1}$, and thus $p_A(\lambda) = \det(\lambda I_n - A) = \det(C)\det(\lambda I_n - B)\det(C)^{-1} = \det(\lambda I_n - B) = p_B(\lambda)$.

T [✓ **F**] If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A and B are the matrices of T with respect to two different bases of $\mathbb{R}^n$, then A is symmetric if and only if B is symmetric.

For a counterexample, take A to be any non-symmetric, diagonalizable matrix, and B its (automatically symmetric) diagonalization. (For example, the matrix A of problem 5, for any $t \neq 1$.)

4. (14 points) Let A be the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, whose characteristic polynomial is $p(\lambda) = (\lambda+1)^2(\lambda-2)$. (You do *not* need to check this formula for $p(\lambda)$.)

   (a) Say why A is diagonalizable, without any calculations. (Hint: you might consider A^T.)

   (3 points) $A^T = A$, so A is symmetric. By the Spectral Theorem, A is diagonalizable. (Another reason is that A has an eigenbasis, by looking ahead to the statement of part (b).)

   (b) Find a matrix P so that $A = P \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1}$.

   (Hint: use the eigenspaces of A to form an eigenbasis for $\mathbb{R}^3$; then you do not need to compute P^{-1} or directly compute the product in the expression above!)

   (7 points) P is a matrix whose columns are vectors of an eigenbasis for A. Therefore, to find P, we need to find three linearly independent eigenvectors of A. By the given information, A has two eigenvalues, -1 and 2. For $\lambda = -1$, the eigenspace is

$$E_{-1} = N\left(\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right).$$

   For $\lambda = 2$, the eigenspace is $E_2 = N\left(\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

   Thus, three independent eigenvectors of A are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and we let $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

   Notes: Many people overlooked that the order of the vectors in the matrix P must correspond to the order of the eigenvalues in the diagonalization given. In this case, we have -1 in the first two diagonal entries, and 2 in the last. Hence, eigenvectors corresponding to -1 must be in the first two columns of P, while an eigenvector corresponding to 2 must be in the last column. Also note that there are a lot of possible answers for P, since there are many choices of bases of eigenvectors.

   (c) Determine, with justification, the definiteness of the quadratic form

$$Q(x,y,z) = 2xy + 2xz + 2yz.$$

   (4 points) The form Q is $\boxed{\text{indefinite}}$.

   Solution 1: We can examine the matrix corresponding to $Q(x,y,z)$. That matrix is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, which is the same as the matrix A. From the information given, we know that A has eigenvalues -1 and 2. It follows that Q is indefinite because A has both positive and negative eigenvalues.

   Solution 2: To show Q is indefinite, it is sufficient to find (x_1, y_1, z_1) and (x_2, y_2, z_2) such that $Q(x_1, y_1, z_1) > 0$, but $Q(x_2, y_2, z_2) < 0$. (For example, $(x_1, y_1, z_1) = (1,1,1)$ and $(x_2, y_2, z_2) = (-1,-1,1)$.) For full credit using this approach, you need to provide explicit examples of such (x,y,z).

5. (10 points) For any real number t, let $A(t)$ be the matrix $\begin{bmatrix} 1 & 1 & 0 \\ t & 1 & 1-t \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Find all values of t such that the characteristic polynomial of $A(t)$ is equal to $\lambda(\lambda-1)(\lambda-2)$.

(5 points) The characteristic polynomial is

$$\begin{aligned} p(\lambda) = \det(\lambda I_3 - A) &= \begin{vmatrix} \lambda-1 & -1 & 0 \\ -t & \lambda-1 & t-1 \\ 0 & -1 & \lambda-1 \end{vmatrix} \quad \text{[expand along first column]} \\ &= (\lambda-1)\begin{vmatrix} \lambda-1 & t-1 \\ -1 & \lambda-1 \end{vmatrix} - (-t)\begin{vmatrix} -1 & 0 \\ -1 & \lambda-1 \end{vmatrix} \\ &= (\lambda-1)\left((\lambda-1)(\lambda-1)+t-1\right) + t(-1)(\lambda-1) \\ &= (\lambda-1)\left((\lambda-1)(\lambda-1)+t-1-t\right) \\ &= (\lambda-1)(\lambda^2-2\lambda+1-1) \\ &= (\lambda-1)\lambda(\lambda-2). \end{aligned}$$

Hence, the characteristic polynomial equals $\lambda(\lambda-1)(\lambda-2)$ for $\boxed{\text{every real number } t}$.

Notes: The most common mistakes we saw were sign errors, the most minor of which led to the loss of 1 to 2 points. For one, when we create $\lambda I_3 - A$, here we need to change the signs of A due to the subtraction. When expanding determinant, there are negative signs in front of alternate 2×2 determinants, and so on.

(b) Find all values of t such that $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of $A(t)$ with eigenvalue 1.

(5 points) Straight from the definition, a nonzero vector $\mathbf{u}$ is an eigenvector of $A(t)$ with eigenvalue λ if $A(t)\mathbf{u} = \lambda\mathbf{u}$. Using $\lambda = 1$ and $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, we find $A(t)\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, so that

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = A(t)\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ t & 1 & 1-t \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2t+1-t \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ t+1 \\ 1 \end{bmatrix}.$$

Hence $t+1=0$, so that $\boxed{t=-1}$.

Notes: Many people attempted to compute $N(I_3 - A)$ (since this leads to a basis of eigenvectors for $\lambda = 1$) by trying to do row reduction for this matrix in terms of the variable t, but got stuck in the middle of the computation. Although this null space method does lead to a solution, the above solution is much more direct.

6. (12 points) Let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ be a 3×3 symmetric matrix, and write Q for the associated quadratic form; that is,

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}, \quad \text{for } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3.$$

(a) Show that for any $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$, $\quad Q(\mathbf{v}+\mathbf{w}) = Q(\mathbf{v}) + 2\mathbf{v}^T A \mathbf{w} + Q(\mathbf{w})$.

(6 points) First,

$$\begin{aligned} Q(\mathbf{v}+\mathbf{w}) = (\mathbf{v}+\mathbf{w})^T A(\mathbf{v}+\mathbf{w}) &= (\mathbf{v}^T + \mathbf{w}^T) A(\mathbf{v}+\mathbf{w}) \\ &= \mathbf{v}^T A(\mathbf{v}+\mathbf{w}) + \mathbf{w}^T A(\mathbf{v}+\mathbf{w}) \\ &= \mathbf{v}^T A\mathbf{v} + \mathbf{v}^T A\mathbf{w} + \mathbf{w}^T A\mathbf{v} + \mathbf{w}^T A\mathbf{w} \\ &= Q(\mathbf{v}) + \mathbf{v}^T A\mathbf{w} + \mathbf{w}^T A\mathbf{v} + Q(\mathbf{w}). \end{aligned}$$

Note that $\mathbf{w}^T$, A, and $\mathbf{v}$ are 1×3, 3×3, and 3×1 matrices, respectively, so that $\mathbf{w}^T A\mathbf{v}$ is a 1×1 matrix, i.e. a number. Therefore it is symmetric and

$$\mathbf{w}^T A\mathbf{v} = (\mathbf{w}^T A\mathbf{v})^T = \mathbf{v}^T A^T (\mathbf{w}^T)^T = \mathbf{v}^T A\mathbf{w},$$

where the last equality follows from the symmetry of A. Therefore,

$$\begin{aligned} Q(\mathbf{v}+\mathbf{w}) &= Q(\mathbf{v}) + \mathbf{v}^T A\mathbf{w} + \mathbf{w}^T A\mathbf{v} + Q(\mathbf{w}) \\ &= Q(\mathbf{v}) + 2\mathbf{v}^T A\mathbf{w} + Q(\mathbf{w}), \qquad \text{as desired.} \end{aligned}$$

Notes: It should be noted that $\mathbf{w}^T A\mathbf{v}$ is in general not equal to $\mathbf{v}^T A\mathbf{w}$ unless A is symmetric. Also, it's possible to do this question by writing out the entries of the vectors $\mathbf{v}$, $\mathbf{w}$ and the matrix A and doing the matrix multiplication explicitly in terms of those entries. However, the computation will be long and it is easy to make a mistake in that way.

(b) Write $DQ(\mathbf{x})$ for the matrix of partial derivatives of $Q(\mathbf{x})$. Show that

$$\mathrm{D}Q(\mathbf{x}) = 2\mathbf{x}^T A.$$

(Hint: you can directly compute each side of this equation in terms of the entries of A and $\mathbf{x}$.)

(6 points)

General notes: Before trying to prove the given formula, we should first try to understand what the two sides of the formula mean: note that Q is the quadratic form defined by A, so it should be regarded as a function from $\mathbb{R}^3$ to $\mathbb{R}$. Hence, on the left hand side, $DQ(\mathbf{x})$ should be a 1×3 matrix, i.e. a row vector, with entries being the first-order partial derivatives of Q. The right hand side is basically the product of a row vector and a square matrix, and so it should be a row vector too.

Solution: We have that

$$Q(\mathbf{x}) = Q(x_1, x_2, x_3) = a\,x_1^2 + d\,x_2^2 + f\,x_3^2 + 2\,b\,x_1\,x_2 + 2\,c\,x_1\,x_3 + 2\,e\,x_2\,x_3.$$

Taking partial derivatives, we get

$$\frac{\partial Q}{\partial x_1} = 2ax_1 + 2bx_2 + 2cx_3, \qquad \frac{\partial Q}{\partial x_2} = 2bx_1 + 2dx_2 + 2ex_3, \qquad \frac{\partial Q}{\partial x_3} = 2cx_1 + 2ex_2 + 2fx_3.$$

Hence,

$$\begin{aligned} DQ(\mathbf{x}) &= \begin{bmatrix} \frac{\partial Q}{\partial x_1} & \frac{\partial Q}{\partial x_2} & \frac{\partial Q}{\partial x_3} \end{bmatrix} \\ &= \begin{bmatrix} 2ax_1 + 2bx_2 + 2cx_3 & 2bx_1 + 2dx_2 + 2ex_3 & 2cx_1 + 2ex_2 + 2fx_3 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \end{bmatrix} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \\ &= 2\mathbf{x}^T A. \end{aligned}$$

Remark (optional): There is a connection between parts (a) and (b), via the limit definition of the derivative for a scalar-valued function $Q(\mathbf{x}) = Q(x_1, x_2, x_3)$. From Definition 3.8 on page 118 of Colley's *Vector Calculus*, $DQ(\mathbf{x})$ is the 1×3 matrix such that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{Q(\mathbf{x} + \mathbf{h}) - Q(\mathbf{x}) - DQ(\mathbf{x})\mathbf{h}}{||\mathbf{h}||} = 0.$$

(See also the discussion on page 120.) But by part (a), for any $\mathbf{x}, \mathbf{h} \in \mathbb{R}^3$,

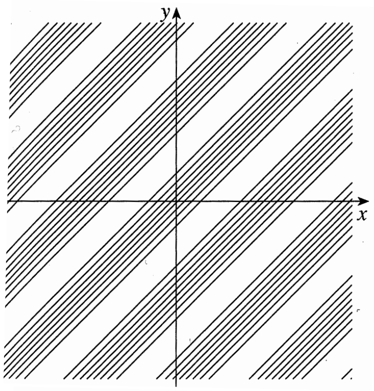
$$Q(\mathbf{x} + \mathbf{h}) = Q(\mathbf{x}) + 2\mathbf{x}^T A\mathbf{h} + Q(\mathbf{h}),$$

and we can argue separately that

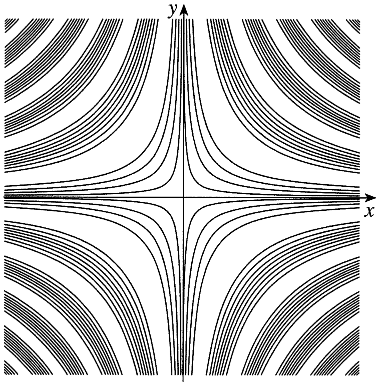
$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{Q(\mathbf{h})}{||\mathbf{h}||} = 0;$$

this allows us to conclude directly (independently of part (b)!) that $DQ(\mathbf{x}) = 2\mathbf{x}^T A$.

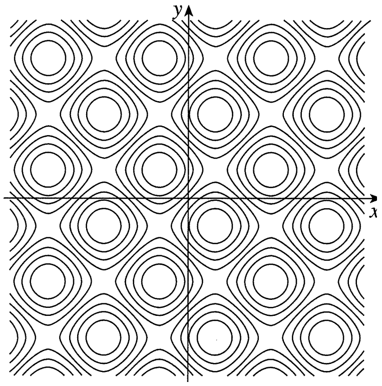
7. (10 points) Match each function below with its set of level curves, chosen from among those labeled I through VI below. No justification is necessary.



I



II



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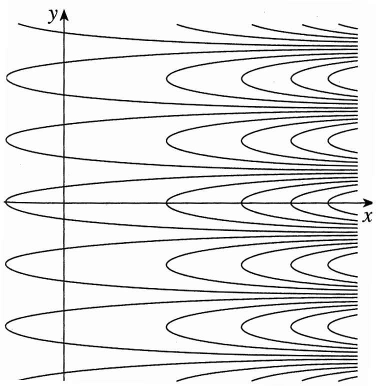

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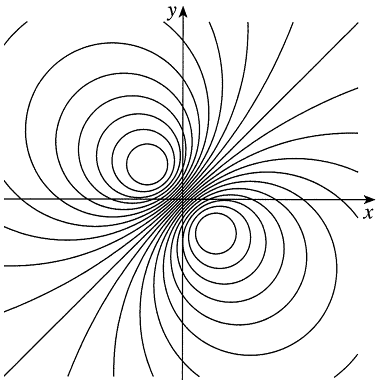
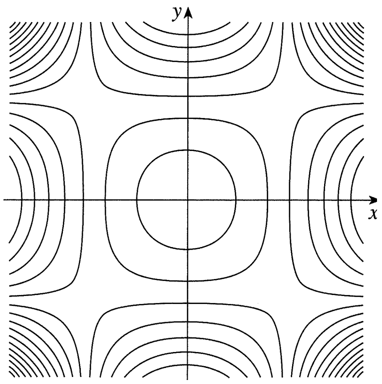


V



VI

Function	I, II, III, IV, V, or VI	Function	I, II, III, IV, V, or VI
$f(x,y) = \sin(xy)$	II	$f(x,y) = e^x \cos y$	IV
$f(x,y) = \sin(x-y)$	I	$f(x,y) = \sin x - \sin y$	III
$f(x,y) = (1-x^2)(1-y^2)$	VI	$f(x,y) = \dfrac{x-y}{1+x^2+y^2}$	V

- Curves of the form $\sin(xy) = C$ have $xy = D$ for constants D; these curves are shown in II.
- If $\sin(x-y) = C$, then $x - y = D$ for some D; these are the curves shown in I.
- If $e^x \cos y = C$, then note x can be written as a function of y; IV is the only remaining set of curves with this property.
- If the curve $\sin x - \sin y = C$ contains some point (a,b), then it also contains all points $(a+2\pi n, b+2\pi n)$; this is the pattern of III.
- Any curve $(1-x^2)(1-y^2) = C$ is symmetric with respect to the x-axis, the y-axis, and the line $y = x$; only VI has all of these symmetries.
- Finally, V by process of elimination! *(Note: this was only one possible approach.)*

8. (15 points) Given the parametrized path $\mathbf{f}(t) = (1 - t^2, t^3 - t)$, for $t \in \mathbb{R}$, which describes a curve in $\mathbb{R}^2$.

   (a) Find all values of t for which $\mathbf{f}'(t)$ is orthogonal to $\mathbf{f}''(t)$.

(5 points) First, $\mathbf{f}'(t) = (-2t, 3t^2 - 1)$ and $\mathbf{f}''(t) = (-2, 6t)$. We want

$$\mathbf{f}'(t) \cdot \mathbf{f}''(t) = 0, \quad \text{so}$$

$$18t^3 - 2t = 0, \quad \text{so} \qquad 2t(9t^2 - 1) = 2t(3t - 1)(3t + 1) = 0.$$

Thus, we obtain three such values for t: $\boxed{0,\ -\frac{1}{3},\ \frac{1}{3}}$.

   (b) Find a point (a, b) in $\mathbb{R}^2$ which lies on the curve for two different values of t; i.e., which satisfies

$$(a, b) = \mathbf{f}(t_1) \text{ and } (a, b) = \mathbf{f}(t_2), \text{ for two values } t_1 \neq t_2.$$

   Find (a, b) and the values t_1 and t_2.

(5 points) The points we are looking for must satisfy

$$1 - t_1^2 = 1 - t_2^2 \quad \text{and} \quad t_1(t_1^2 - 1) = t_2(t_2^2 - 1).$$

Since the quantities in the parenthesis of the second expression are equal by our first equation, and we don't want t_1 to be the same as t_2, we conclude that

$$1 - t_1^2 = 1 - t_2^2 = 0.$$

Thus, t_1 and t_2 are solutions to $1 - t^2 = 0$, namely $t_1 = 1$, $t_2 = -1$. We find $\boxed{\mathbf{f}(1) = \mathbf{f}(-1) = (0, 0)}$.

   (c) Describe the points (x, y) lying in the range of $\mathbf{f}$; that is, give an equation in x and y alone satisfied by the points on the curve.

(5 points)

Solution 1: We have

$$x = 1 - t^2 \quad \text{and} \quad y = t^3 - t = t(t^2 - 1),$$

so $y = -tx$, and thus $t = -y/x$. Substituting this into the expression for x, we find

$$x = 1 - \left(-\frac{y}{x}\right)^2, \quad \text{or} \quad \boxed{y^2 = x^2 - x^3}.$$

Solution 2: Solving $x = 1 - t^2$ for t, we have

$$t^2 = 1 - x, \quad \text{so} \quad t = \pm\sqrt{1 - x}.$$

Substituting this into the expression for y, we thus find

$$y = t^3 - t = t(t^2 - 1) = -tx = \pm x\sqrt{1 - x}.$$

(Technically this is two equations in x and y, giving two different pieces of the curve, but we accepted this for full credit. Squaring both sides yields the single equation of the first solution.)