Math 51 Exam 2 — May 20, 2008

Name : ____

Section Leader: (Circle one)	Fai Chandee	Joseph Cheng	David Fernandez-Duque	Anca Vacarescu	Bezirgen Veliyev

Section Time: (Circle one)	10:00	11:00	1:15	2:15

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- **You have 90 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner. You have only until **Tuesday, June 3**, to resubmit your exam for any regrade considerations; consult your section leader about the exact details of the submission process.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	8		**5**	10	
2	15		**6**	12	
3	16		**7**	10	
4	14		**8**	15	
			Total	100	

1. (8 points) Compute the following determinant:

$$\begin{vmatrix} 2 & 4 & -2 & -1 \\ 1 & 1 & -1 & 0 \\ 3 & 2 & -1 & -1 \\ 2 & 2 & -3 & 1 \end{vmatrix}$$

2. (15 points) Let $\mathcal{B}$ be the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

(You don't have to check that $\mathcal{B}$ is a basis.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by the following formulas:

$$T(\mathbf{v}_1) = \mathbf{v}_2, \quad T(\mathbf{v}_2) = \mathbf{v}_3, \quad T(\mathbf{v}_3) = \mathbf{v}_1.$$

(a) Find the matrix of T with respect to the basis $\mathcal{B}$.

(b) Find the matrix of T with respect to the standard basis.

(c) Find a matrix X such that

$$X \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(Hint: is there a relation between X and T^2?)

3. (16 points) Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

T **F** If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ is also a basis for $\mathbb{R}^n$, then T must be invertible.

T **F** It is possible to find a 3×2 matrix A and a 2×3 matrix B, satisfying $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

T **F** It is possible to find a 2×3 matrix C and a 3×2 matrix D, satisfying $CD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

T **F** Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with matrix A, such that whenever R is a region in $\mathbb{R}^2$, then the area of $T(R)$ is equal to the area of R. It follows that $\det(A) = 1$.

T **F** If $\mathcal{B}$ is an orthonormal basis for $\mathbb{R}^n$, and C is its associated change-of-basis matrix, then it follows that C is a symmetric matrix.

T **F** If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A and B are the matrices of T with respect to two different bases of $\mathbb{R}^n$, then $\det(A) = \det(B)$.

T **F** If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A and B are the matrices of T with respect to two different bases of $\mathbb{R}^n$, then A and B have the same characteristic polynomial.

T **F** If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A and B are the matrices of T with respect to two different bases of $\mathbb{R}^n$, then A is symmetric if and only if B is symmetric.

4. (14 points) Let A be the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, whose characteristic polynomial is $p(\lambda) = (\lambda+1)^2(\lambda-2)$. (You do *not* need to check this formula for $p(\lambda)$.)

(a) Say why A is diagonalizable, without any calculations. (Hint: you might consider A^T.)

(b) Find a matrix P so that $A = P \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1}$.

(Hint: use the eigenspaces of A to form an eigenbasis for $\mathbb{R}^3$; then you do not need to compute P^{-1} or directly compute the product in the expression above!)

(c) Determine, with justification, the definiteness of the quadratic form

$$Q(x, y, z) = 2xy + 2xz + 2yz.$$

5. (10 points) For any real number t, let $A(t)$ be the matrix $\begin{bmatrix} 1 & 1 & 0 \\ t & 1 & 1-t \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Find all values of t such that the characteristic polynomial of $A(t)$ is equal to $\lambda(\lambda-1)(\lambda-2)$.

(b) Find all values of t such that $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of $A(t)$ with eigenvalue 1.

6. (12 points) Let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ be a 3×3 symmetric matrix, and write Q for the associated quadratic form; that is,

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}, \quad \text{for } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3.$$

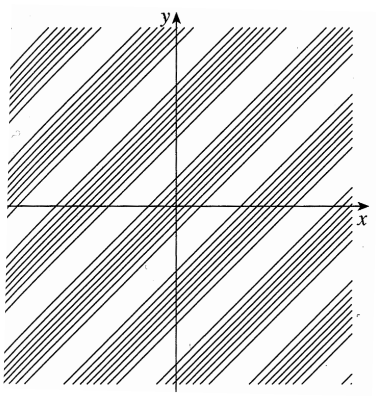
(a) Show that for any $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$, $\quad Q(\mathbf{v} + \mathbf{w}) = Q(\mathbf{v}) + 2\mathbf{v}^T A \mathbf{w} + Q(\mathbf{w})$.

(b) Write $DQ(\mathbf{x})$ for the matrix of partial derivatives of $Q(\mathbf{x})$. Show that

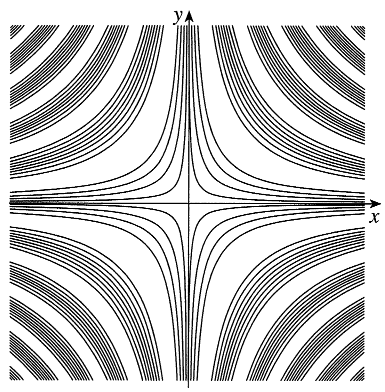
$$\mathrm{D}Q(\mathbf{x}) = 2\mathbf{x}^T A.$$

(Hint: you can directly compute each side of this equation in terms of the entries of A and $\mathbf{x}$.)

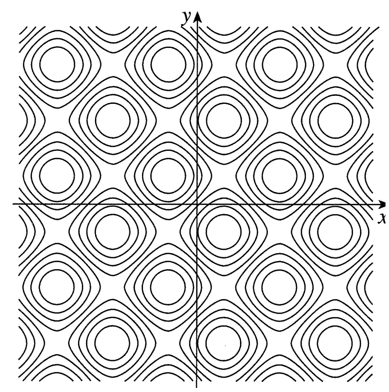
7. (10 points) Match each function below with its set of level curves, chosen from among those labeled I through VI below. No justification is necessary.



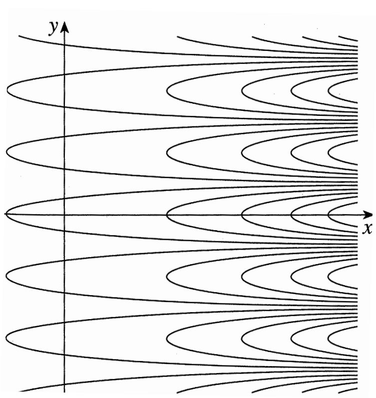
I



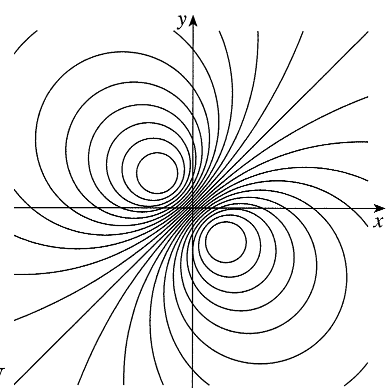
II



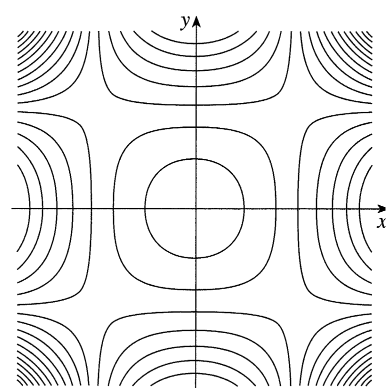
III



IV



V



VI

Function	I, II, III, IV, V, or VI	Function	I, II, III, IV, V, or VI
$f(x,y) = \sin(xy)$		$f(x,y) = e^x \cos y$	
$f(x,y) = \sin(x-y)$		$f(x,y) = \sin x - \sin y$	
$f(x,y) = (1-x^2)(1-y^2)$		$f(x,y) = \dfrac{x-y}{1+x^2+y^2}$	

8. (15 points) Given the parametrized path $\mathbf{f}(t) = (1 - t^2, t^3 - t)$, for $t \in \mathbb{R}$, which describes a curve in $\mathbb{R}^2$.

(a) Find all values of t for which $\mathbf{f}'(t)$ is orthogonal to $\mathbf{f}''(t)$.

(b) Find a point (a, b) in $\mathbb{R}^2$ which lies on the curve for two different values of t; i.e., which satisfies

$$(a, b) = \mathbf{f}(t_1) \text{ and } (a, b) = \mathbf{f}(t_2), \text{ for two values } t_1 \neq t_2.$$

Find (a, b) and the values t_1 and t_2.

(c) Describe the points (x, y) lying in the range of $\mathbf{f}$; that is, give an equation in x and y alone satisfied by the points on the curve.