Math 51 Exam 1 Solutions — April 22, 2008

1. (10 points) Compute, showing all steps, the reduced row echelon form of the matrix

$$\begin{bmatrix} 0 & 2 & -1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 2 & -1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{matrix} swap \\ swap \\ \\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 2 & -1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{matrix} \\ \\ -I \\ -I \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{matrix} \\ (\frac{1}{2}) \\ \\ \\ \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{matrix} -II \\ \\ \\ \\ \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 5/2 & 7/2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{matrix} -\frac{5}{2}III \\ +\frac{1}{2}III \\ \\ +III \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. (15 points) Consider the matrix A below, and its reduced row echelon form (which you *do not* have to verify!):

$$A = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}, \quad \texttt{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the null space of A. You do not need to prove that your collection is a basis.

(5 points; more than one answer is possible, but this is the most direct solution.) From $\texttt{rref}(A)$, we see that the system $A\mathbf{x} = \mathbf{0}$ has been reduced to:

$$x_1 + 2x_4 = 0,$$
$$x_3 + x_4 = 0.$$

Thus a solution vector $\mathbf{x}$ can be expressed: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$,

so that $N(A)$ is the span of the vectors $\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. As is always true for vectors obtained through this method, these vectors are linearly independent, which can be seen directly by considering the second and fourth components. Thus, a basis for $N(A)$ is

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Notes: Many people incorrectly took $x_2 = 0$ because there is no x_2 in any of the initial equations; this means they did not get $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ in the null space.

(b) Are the columns of A linearly independent? If so, explain why; if not, write a non-trivial linear relation they satisfy.

(2 points) No, the columns of A are linearly dependent, as seen by the presence of columns in $\texttt{rref}(A)$ that do not contain pivots. If we denote the columns of A by $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$, then two relations that we can easily write down are:

$$\mathbf{v_4} = 2\mathbf{v_1} + \mathbf{v_3}, \quad \text{or}$$

$$0\mathbf{v_1} + 1\mathbf{v_2} + 0\mathbf{v_3} + 0\mathbf{v_4} = \mathbf{0}.$$

(ctd. from previous page) $A = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$, $\texttt{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) Find a basis for the column space of A. You do not need to prove that your collection is a basis.

(3 points) There are pivots in the first and third columns of $\texttt{rref}(A)$, so the corresponding columns in the initial matrix A are linearly independent. (Furthermore, by part (b) or otherwise, the second and fourth columns of A can be expressed as linear combinations of the first and third.) Hence,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $C(A)$.

(d) Give a specific vector $\mathbf{b} \in \mathbb{R}^4$ such that the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution $\mathbf{x} \in \mathbb{R}^4$ (and give this solution), or state why such a $\mathbf{b}$ does not exist.

(5 points) Such a $\mathbf{b}$ does not exist for this matrix A. In fact, for any $\mathbf{b}$ in $\mathbb{R}^4$, the system $A\mathbf{x} = \mathbf{b}$ has either no solutions, or infinitely many solutions. (It's possible for there to be no solutions because $\dim(C(A)) = 2 < 4$.) If we assume for some $\mathbf{b}$ that there exists a particular solution $\mathbf{x_p}$, then any vector $\mathbf{x_p} + \mathbf{x_h}$, for $\mathbf{x_h}$ in the null space of A, is also a solution. But since $\dim(N(A)) = 2$, this would yield infinitely many solutions. In any case, there is no $\mathbf{b}$ that gives a system having a unique solution.

Notes: Many people said that since $\texttt{rref}(A)$ has a free variable, we get infinitely many solutions. This is incomplete, we need to remember that there might be no solutions as well.

3. (14 points) In each of the parts below, a set of vectors in $\mathbb{R}^3$ is specified. In each case, find an expression for this set in *parametric form*, using linearly independent vectors. (Alternatively, if applicable, you may find a basis for the set.) Show the steps of your computations.

(a) $S = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_2 + x_3 = 1 \text{ and} \\ 2x_1 - x_2 + x_3 = -1 \end{array} \right\}.$

(7 points) Via row operations, we find

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & -1 \end{array}\right] -2I \quad \rightsquigarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & -3 \end{array}\right] (-\tfrac{1}{3})$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} & 1 \end{array}\right] -II \quad \rightsquigarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 1 \end{array}\right].$$

Hence the system reduces to: $x_1 + \frac{2}{3}x_3 = 0$, $x_2 + \frac{1}{3}x_3 = 1$. So x_3 is a free variable, and the system above has infinitely many solutions, which will be of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$.

Thus, the parametric representation is: $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbf{R} \right\}.$

Notes: Although S is a line in $\mathbb{R}^3$, S is not a subspace (why?), so it does not have a basis.

(b) Y is the set of all vectors in $\mathbb{R}^3$ orthogonal to $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. (Hint: you can approach this by solving a system.)

(7 points) Y is the set of vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that satisfy: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 0,$

which can be written as $2x_1 + x_2 + 3x_3 = 0$, a "system" of one equation, three unknowns. Setting up the coefficient matrix and reducing to `rref`, we obtain:

$$\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

From this we can see that we have one pivot variable and two free variables, and the solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 - \frac{3}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the parametric representation is: $Y = \left\{ t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \;\middle|\; t, s \in \mathbf{R} \right\}.$

Notes: Alternatively, Y may be viewed as the span of the two basis vectors $\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$.

4. (20 points) Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

(2 points each)

T [**F**] For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^{k+1}$, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.

Independence is never automatic; the above set can only be linearly independent if none of the vectors can be written as a linear combination of the remaining ones.

[**T**] **F** For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^{k+1}$, the set $\{0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent.

Any set containing the zero vector is automatically dependent; here, $\mathbf{0}$ may be written as the linear combination $0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k$.

[**T**] **F** If the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, then there is at least one free variable.

For this system, at least one column of $\texttt{rref}(A)$ lacks a pivot, and corresponds to a free variable.

T [**F**] The null space of $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ is span $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right)$.

The null space also includes the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, which is not in the set given above.

[**T**] **F** If B is a 3-by-3 matrix such that the linear transformation $T(\mathbf{x}) = B\mathbf{x}$ defines a projection of $\mathbf{x} \in \mathbb{R}^3$ onto one of the coordinate axes of $\mathbb{R}^3$, then the column space of B has dimension 1.

$C(B)$ is the set of possible products $B\mathbf{x}$; that is, the set of possible outputs of T. But projection onto any line (not just onto one of the axes) yields a one-dimensional set of possible outcomes.

T [**F**] The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2 \\ 2x_2 \end{bmatrix}$ is a linear transformation.

For example, $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ but $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, meaning $T\left(2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \neq 2\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

[**T**] **F** If Q is any square matrix, then the null space of Q has dimension equal to the number of zero rows in the reduced row echelon form of Q.

If Q is n-by-n, and $R = \texttt{rref}(Q)$, then $\dim(N(Q)) = (\#\text{cols of } R \text{ without a pivot}) =$
$n - (\#\text{cols of } R \text{ with a pivot}) = n - (\#\text{rows of } R \text{ with a pivot}) = (\#\text{zero rows in } R)$.

[**T**] **F** There exists a subspace of $\mathbb{R}^6$ of dimension 5.

Consider the span of any 5 independent vectors (such as any 5 of the 6 standard basis vectors).

T [**F**] The matrix $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is in reduced row echelon form.

Neither row contains a pivot entry, since a pivot must be the only nonzero entry in its column.

T [**F**] For any vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, $||\mathbf{x} + \mathbf{y}|| = ||\mathbf{x}|| + ||\mathbf{y}||$.

For example, compare the left and right sides when $\mathbf{x}$ is any nonzero vector and $\mathbf{y} = -\mathbf{x}$.

5. (14 points) Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ containing the points $(-1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

(a) Find an equation of the plane $\mathcal{P}$; give your answer as a single linear equation involving coordinates x, y, and z.

(7 points) Vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

are parallel to the plane. Therefore, their cross product

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} 6 \\ -3 \\ -2 \end{bmatrix}$$

is a vector normal to the plane. So the equation of the plane is:

$$6(x+1) - 3y - 2z = 0,$$

i.e.,

$$6x - 3y - 2z = -6.$$

(b) Find values (a, b, c) satisfying both of the following conditions simultaneously:

- the *point* $M = (a, b, c)$ lies on the plane $\mathcal{P}$, and
- the *vector* $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is normal (perpendicular) to the plane $\mathcal{P}$.

(7 points) Vector $\mathbf{v}$, being perpendicular to the plane $\mathcal{P}$, must be parallel to the normal vector we found in part (a), i.e.,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{v} = s \begin{bmatrix} 6 \\ -3 \\ -2 \end{bmatrix}$$

for some scalar s. In other words, $a = 6s$, $b = -3s$, and $c = -2s$. It remains only to find s. However, the point $M = (a, b, c)$ belongs to the plane $\mathcal{P}$, so $6a - 3b - 2c = -6$, i.e.,

$$6(6s) - 3(-3s) - 2(-2s) = -6,$$

so

$$49s = -6, \text{ and thus, } s = -6/49.$$

It follows that the point M has coordinates: $\left(-\frac{36}{49}, \frac{18}{49}, \frac{12}{49}\right)$.

6. (15 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^6$ be the function defined by: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_2 \\ x_4 \\ x_3 \\ 0 \\ 0 \end{bmatrix}$.

(a) Is T a linear transformation? If so, give the matrix A associated to T; if not, explain why.

(7 points) Yes, T is a linear transformation. There are two ways to see this.

First method: Check that T satisfies the criteria of a linear transformation.

i. $$T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \\ x_4+y_4 \end{bmatrix}\right) = \begin{bmatrix} x_3+y_3 \\ x_2+y_2 \\ x_4+y_4 \\ x_3+y_3 \\ 0+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ x_4 \\ x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} y_3 \\ y_2 \\ y_4 \\ y_3 \\ 0 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}\right).$$

ii. For any scalar c, $$T\left(\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \\ cx_4 \end{bmatrix}\right) = \begin{bmatrix} cx_3 \\ cx_2 \\ cx_4 \\ cx_3 \\ 0 \\ 0 \end{bmatrix} = c\begin{bmatrix} x_3 \\ x_2 \\ x_4 \\ x_3 \\ 0 \\ 0 \end{bmatrix} = cT\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right).$$

Second method: We can find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. By Proposition 13.2,

$$A = \begin{bmatrix} | & | & | & | \\ T(\mathbf{e_1}) & T(\mathbf{e_2}) & T(\mathbf{e_3}) & T(\mathbf{e_4}) \\ | & | & | & | \end{bmatrix},$$

where $T(\mathbf{e_1}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $T(\mathbf{e_2}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $T(\mathbf{e_3}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, and $T(\mathbf{e_4}) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Thus, $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Now, you can verify through multiplication that $T(\mathbf{x}) = A\mathbf{x}$, and so T is linear. (We gave full credit for simply giving the correct matrix A, since the matrix multiplication is so simple here.)

Notes: A common mistake was to say A was a 4-by-4 matrix, rather than 6-by-4. Also, many people said T is not a linear transformation because the dimension of vectors in the image is 6 while the dimension of vectors in the domain is 4, but this issue is not relevant to whether T is linear.

(b) Determine all vectors $\mathbf{x}$ in $\mathbb{R}^4$ such that $T(\mathbf{x}) = \mathbf{0}$.

(4 points) The answer is span $\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \left\{ \begin{bmatrix}t\\0\\0\\0\end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$. Note that directly from the formula for T, if $T(\mathbf{x}) = 0$, then $x_2 = x_3 = x_4 = 0$, but the first component x_1 can be anything.

Alternatively, this question asks for the null space of matrix A (or, the *kernel* of T). You can find the reduced row echelon form for A, and you get

$$\texttt{rref}(A) = \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

From $\texttt{rref}(A)$, x_1 is a free variable, and $x_2 = x_3 = x_4 = 0$, as before.

Notes: Although many people found $\texttt{rref}(A)$ and said that $x_2 = x_3 = x_4 = 0$, many forgot that x_1 can be anything.

(c) Let W be the set of vectors in $\mathbb{R}^6$ which can be expressed as $T(\mathbf{x})$ for an appropriate $\mathbf{x} \in \mathbb{R}^4$. It is a fact that for this T, the set W is a subspace of $\mathbb{R}^6$. Find a set of vectors that spans W.

(4 points) Since we are told that W is a subspace, all we need to do is to find a set of vectors that span W. This is the same thing as finding a set of vectors that span the column space of A (or equivalently, the *image* of T). We can take the columns of A as our answer here, since we're not asked for a basis. Or you can find a basis of A, by deleting the first column of A (the one that corresponds to the free variable x_1). Thus, two answers to this question are

$$\mathcal{B}_1 = \left\{ \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} \right\}, \quad \text{or} \quad \mathcal{B}_2 = \left\{ \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} \right\}.$$

Notes: It was not required to prove that W is a subspace here, though the easiest way to see it is to realize it directly as a span of vectors: the formula for T implies that, in parametric form,

$$W = \left\{ \begin{bmatrix}x_3\\x_2\\x_4\\x_3\\0\\0\end{bmatrix} \;\middle|\; x_2, x_3, x_4 \in \mathbb{R} \right\} = \left\{ x_2 \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix} + x_3 \begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix} + x_4 \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} \;\middle|\; x_2, x_3, x_4 \in \mathbb{R} \right\},$$

the latter set clearly being the span of the three nonzero vectors in the above set $\mathcal{B}_2$.

7. (12 points)

(a) Complete the sentence: A set V of vectors in $\mathbb{R}^n$ is a subspace if

(6 points) ... V contains $\mathbf{0}$ and is closed under addition and scalar multiplication.

ALSO ACCEPTABLE:

... all three of the following conditions on V are true:

- the zero vector lies in V;
- for any $\mathbf{x}$, $\mathbf{y} \in V$, the vector $\mathbf{x} + \mathbf{y}$ also lies in V;
- for any $\mathbf{x} \in V$ and any scalar c, the vector $c\mathbf{x}$ also lies in V.

Notes: These 6 points were intended to be easily taken, by giving the definition — words which, by the way, do not always seem to be understood, so please refer back to your notes! The main confusion was to mistake a *subspace* for a *linear map*. We saw often arguments like "*V is a subspace if* $V(\mathbf{x}+\mathbf{y}) = V(\mathbf{x})+V(\mathbf{y})$," which does not make any sense, as a subspace is a *collection* of vectors, and a linear map is a *transformation* (function) of vectors.

(b) If S is a set of vectors in $\mathbb{R}^n$, let $S^\perp$ be the set of vectors $\mathbf{v}$ in $\mathbb{R}^n$ satisfying $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{w}$ in S. Show that $S^\perp$ is a subspace of $\mathbb{R}^n$.

(6 points) **General notes:**

- This was the most abstract part of the midterm, as S was supposed arbitrary. Due to the difficulty level, one needs to proceed in order, especially by **introducing the notations**, so that the grader does not have to *guess* if $\mathbf{x}$ is in S or in $S^\perp$, etc. Anyway, it is not just for the grader, but also for the student: clarity and success in mathematics go together!
- The question was to show that $S^\perp$ is a subspace, whereas S is not supposed to be a subspace.
- $S^\perp$ may be small or big; for example $(\mathbb{R}^n)^\perp = 0$, but we saw that in $\mathbb{R}^3$, if S is a line, then $S^\perp$ is a plane. In fact, $S^\perp$ gets smaller as S gets bigger.

1st solution: (This is the solution expected from you.) One has to check that the three conditions to be a subspace hold for $S^\perp$.

- $\boxed{\mathbf{0} \in S^\perp}$ One has to check if $\mathbf{0} \cdot \mathbf{s} = 0$, for all $\mathbf{s}$ in S. But $\mathbf{0} \cdot \mathbf{x} = 0$ for any $\mathbf{x} \in \mathbb{R}^n$, so that is true for $\mathbf{x} = \mathbf{s} \in S$.
- $\boxed{S^\perp \text{ is closed under addition}}$ Let $\mathbf{x}, \mathbf{y}$ be in $S^\perp$, let's check if $\mathbf{x} + \mathbf{y}$ is in $S^\perp$ too. Let $\mathbf{s} \in S$, one has:
$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{s} = \mathbf{x} \cdot \mathbf{s} + \mathbf{y} \cdot \mathbf{s} = 0 + 0 = 0$$
This holds for any $\mathbf{s} \in S$, so $\mathbf{x} + \mathbf{y} \in S^\perp$.
- $\boxed{S^\perp \text{ is closed under scalar multiplication}}$ Let $\mathbf{x}$ be in $S^\perp$, $c \in \mathbb{R}$, and let's show that $c\mathbf{x}$ is in $S^\perp$. Let $\mathbf{s} \in S$, one has:
$$(c\mathbf{x}) \cdot \mathbf{s} = c(\mathbf{x} \cdot \mathbf{s}) = c \cdot 0 = 0.$$
As $\mathbf{s}$ is arbitrary, $S^\perp$ is closed under scalar multiplication.

This shows that $S^\perp$ is a subspace of $\mathbb{R}^n$.

We propose two other solutions, which suggest deeper connections with other topics from the course, but which are also more abstract.

2nd solution: This one has to do with null spaces. For $\mathbf{s} \in S$ consider the following map:

$$T_{\mathbf{s}} : \begin{cases} \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \mathbf{x} & \longmapsto & T_s(\mathbf{x}) = \mathbf{x} \cdot \mathbf{s} \end{cases}$$

The properties of the dot product make sure that this map is linear. This is not needed, but the reader may check that the matrix associated to $T_{\mathbf{s}}$ is the 1-by-n matrix $A_{\mathbf{s}} = [s_1 \ s_2 \ \dots \ s_n]$.
By definition, one has:

$$\{\mathbf{s}\}^{\perp} = N(A_{\mathbf{s}}) = \ker(T_{\mathbf{s}}).$$

But $S^{\perp}$ is the set of vectors orthogonal to *all* the $\mathbf{s}$, so:

$$S^{\perp} = \bigcap_{\mathbf{s}\in S} \{\mathbf{s}\}^{\perp} = \bigcap_{\mathbf{s}\in S} N(A_{\mathbf{s}}) = \bigcap_{\mathbf{s}\in S} \ker(T_{\mathbf{s}}).$$

Each $N(A_{\mathbf{s}})$ is a subspace, and so is their intersection (cf. Ex. 10.21 of Levandovsky).

3rd solution: A variant from the preceding. Consider the subspace spanned by S:

$$V = \operatorname{span}(\mathbf{s} \,|\, \mathbf{s} \in S)$$

V is a subspace of $\mathbb{R}^n$, so $\dim(V) = d \leq n$. One sees that (if you don't: exercise!):

$$S^{\perp} = V^{\perp}.$$

So, choose a basis $\mathcal{B} = \{\mathbf{s}_1, \dots, \mathbf{s}_d\}$ of V, whose elements may be taken in S (why?). Then, one has:

$$S^{\perp} = V^{\perp} = \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{x} \cdot \mathbf{s}_i = 0, \text{ for all } i = 1, \dots, d\}$$

which means that if T is the following map:

$$T: \begin{cases} \mathbb{R}^n & \longrightarrow & \mathbb{R}^d \\ \mathbf{x} & \longmapsto & T(\mathbf{x}) = \begin{bmatrix} \mathbf{x} \cdot \mathbf{s}_1 \\ \vdots \\ \mathbf{x} \cdot \mathbf{s}_d \end{bmatrix} \end{cases}$$

whose associated matrix is:

$$A = \begin{bmatrix} \text{—} \mathbf{s}_1^T \text{—} \\ \vdots \\ \text{—} \mathbf{s}_d^T \text{—} \end{bmatrix}$$

then:

$$S^{\perp} = V^{\perp} = N(A) = \ker(T).$$

A null space is a subspace, as seen in class; thus, so is $S^{\perp}$.