# Math 51 Exam 1 — April 22, 2008

**Name** : ________________________________________________

| **Section Leader:** (Circle one) | Fai Chandee | Joseph Cheng | David Fernandez-Duque | Anca Vacarescu | Bezirgen Veliyev |
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| **Section Time:** (Circle one) | 10:00 | 11:00 | 1:15 | 2:15 |
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- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- **You have 90 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner. You have only until **Tuesday, May 6**, to resubmit your exam for any regrade considerations; consult your section leader about the exact details of the submission process.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

**Signature:** ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

| | | | | | |
|---|---|---|---|---|---|
| **1** | 10 | | **5** | 14 | |
| **2** | 15 | | **6** | 15 | |
| **3** | 14 | | **7** | 12 | |
| **4** | 20 | | **Total** | 100 | |

1. (10 points) Compute, showing all steps, the reduced row echelon form of the matrix

$$\begin{bmatrix} 0 & 2 & -1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

2. (15 points) Consider the matrix $A$ below, and its reduced row echelon form (which you *do not* have to verify!):

$$A = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}, \quad \texttt{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the null space of $A$. You do not need to prove that your collection is a basis.

(b) Are the columns of $A$ linearly independent? If so, explain why; if not, write a non-trivial linear relation they satisfy.

(ctd. from previous page) $A = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$, $\texttt{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) Find a basis for the column space of $A$. You do not need to prove that your collection is a basis.

(d) Give a specific vector $\mathbf{b} \in \mathbb{R}^4$ such that the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution $\mathbf{x} \in \mathbb{R}^4$ (and give this solution), or state why such a $\mathbf{b}$ does not exist.

3. (14 points) In each of the parts below, a set of vectors in $\mathbb{R}^3$ is specified. In each case, find an expression for this set in *parametric form*, using linearly independent vectors. (Alternatively, if applicable, you may find a basis for the set.) Show the steps of your computations.

   (a) $S = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x_1 + x_2 + x_3 = 1 \text{ and} \\ 2x_1 - x_2 + x_3 = -1 \end{matrix} \right\}.$

   (b) $Y$ is the set of all vectors in $\mathbb{R}^3$ orthogonal to $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. (Hint: you can approach this by solving a system.)

4. (20 points) Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

**T** **F** For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^{k+1}$, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.

**T** **F** For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^{k+1}$, the set $\{0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent.

**T** **F** If the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, then there is at least one free variable.

**T** **F** The null space of $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ is span $\left( \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right)$.

**T** **F** If $B$ is a 3-by-3 matrix such that the linear transformation $T(\mathbf{x}) = B\mathbf{x}$ defines a projection of $\mathbf{x} \in \mathbb{R}^3$ onto one of the coordinate axes of $\mathbb{R}^3$, then the column space of $B$ has dimension 1.

**T** **F** The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2 \\ 2x_2 \end{bmatrix}$ is a linear transformation.

**T** **F** If $Q$ is any square matrix, then the null space of $Q$ has dimension equal to the number of zero rows in the reduced row echelon form of $Q$.

**T** **F** There exists a subspace of $\mathbb{R}^6$ of dimension 5.

**T** **F** The matrix $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is in reduced row echelon form.

**T** **F** For any vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, $||\mathbf{x} + \mathbf{y}|| = ||\mathbf{x}|| + ||\mathbf{y}||$.

5. (14 points) Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ containing the points $(-1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

   (a) Find an equation of the plane $\mathcal{P}$; give your answer as a single linear equation involving coordinates $x$, $y$, and $z$.

   (b) Find values $(a, b, c)$ satisfying both of the following conditions simultaneously:

   - the *point* $M = (a, b, c)$ lies on the plane $\mathcal{P}$, and
   - the *vector* $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is normal (perpendicular) to the plane $\mathcal{P}$.

6. (15 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^6$ be the function defined by: $$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_2 \\ x_4 \\ x_3 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Is $T$ a linear transformation? If so, give the matrix $A$ associated to $T$; if not, explain why.

(b) Determine all vectors $\mathbf{x}$ in $\mathbb{R}^4$ such that $T(\mathbf{x}) = \mathbf{0}$.

(c) Let $W$ be the set of vectors in $\mathbb{R}^6$ which can be expressed as $T(\mathbf{x})$ for an appropriate $\mathbf{x} \in \mathbb{R}^4$. It is a fact that for this $T$, the set $W$ is a subspace of $\mathbb{R}^6$. Find a set of vectors that spans $W$.

7. (12 points)

   (a) Complete the sentence: A set $V$ of vectors in $\mathbb{R}^n$ is a subspace if

   (b) If $S$ is a set of vectors in $\mathbb{R}^n$, let $S^\perp$ be the set of vectors $\mathbf{v}$ in $\mathbb{R}^n$ satisfying $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{w}$ in $S$. Show that $S^\perp$ is a subspace of $\mathbb{R}^n$.