MATH 51 MIDTERM 2 SOLUTIONS

November 13, 2008

1(a). Find the matrix for for the linear map T given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 7x - 2y \\ x + 3y \\ 5y \end{bmatrix}$.

Solution: By inspection, matrix is $\begin{bmatrix} 7 & -2 \\ 1 & 3 \\ 0 & 5 \end{bmatrix}$. Alternatively, one can find the matrix by calculating $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ to get columns 1 and 2, respectively.

1(b). Find the matrix for reflection in $\mathbf{R}^2$ about the line $y = -x$.

Solution: First column is $T(\mathbf{e}_1) = -\mathbf{e}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Second column is $T(\mathbf{e}_2) = -\mathbf{e}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Thus the matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

1(c). Find the matrix for $T : \mathbf{R}^3 \to \mathbf{R}^3$, where T is rotation by 180° about the y-axis, followed by rotation by 180° about the z-axis.

Solution:

$$\mathbf{e}_1 \to -\mathbf{e}_1 \to \mathbf{e}_1$$
$$\mathbf{e}_2 \to \mathbf{e}_2 \to -\mathbf{e}_2$$
$$\mathbf{e}_3 \to -\mathbf{e}_3 \to -\mathbf{e}_3,$$

so the columns of the matrix are $\mathbf{e}_1$, $-\mathbf{e}_2$, and $-\mathbf{e}_3$. Thus the matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2. Find the determinant of the matrix $C = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Solution: $\begin{vmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & -3 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ -3 & -5 \end{vmatrix} = 1(-5) - 4(-3) = 7$, where we added row 1 to row 2 and subtracted 2 times row 1 from row 3, and then expanded using the first column. Instead of expanding using the first column we could also add 3 times row 2 to row 3:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & -3 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 7 \end{vmatrix} = 1 \cdot 1 \cdot 7 = 7.$$

3. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 0 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 1 & | & 0 & 0 & 1 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 0 & | & -1 & 0 & 1 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & | & 0 & -1 & 1 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix}$$

so A^{-1} is $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

4(a). Find all eigenvalues of the matrix $A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 1 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

Solution:

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -4 & -3 \\ -1 & \lambda - 1 & -2 \\ 0 & 0 & \lambda - 7 \end{vmatrix} = (\lambda - 7) \begin{vmatrix} \lambda - 1 & -4 \\ -1 & \lambda - 1 \end{vmatrix}$$
$$= (\lambda - 7)((\lambda - 1)^2 - (-1)(-4)) = (\lambda - 7)((\lambda - 1)^2 - 4)$$

Thus $\lambda = 7$ or $(\lambda - 1)^2 = 4$. In the latter case, $\lambda - 1 = \pm 2$, so $\lambda = 1 \pm 2$, so $\lambda = 3$ or $\lambda = -1$. Thus the eigenvalues are 7, 3, and -1.

4(b). The matrix $B = \begin{bmatrix} -1 & 8 \\ 8 & 11 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = (\lambda - 15)(\lambda + 5)$. Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of B.

Solution: For $\lambda = 15$, we find a nonzero vector in the nullspace of $(15\, I - B)\mathbf{v} = \mathbf{0}$. Note that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is such a vector. (Any nonzero scalar multiple of $\mathbf{v}_1$ would also do.)

We can get the second basis eigenvector in either of two ways:

(1) By symmetry of B, we know any eigenvector with eigenvalue -5 must be orthogonal to $\mathbf{v}_1$. Thus we can just rotate $\mathbf{v}_1$ by $90°$ to get our second eigenvector: $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(2) For $\lambda = -5$, we find a nonzero vector in the nullspace of $-5\, I - B = \begin{bmatrix} -4 & -8 \\ -8 & -16 \end{bmatrix}$. Note that $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is such a vector.

4(c). Suppose that C is a **symmetric** 2x2 matrix with determinant 7. Suppose that the vector $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector of C with eigenvalue 5. Find an eigenvector $\mathbf{w}$ that is **not** a scalar multiple of $\mathbf{v}$, and find its eigenvalue. Explain.

Solution: By symmetry of C , we know that C must have an orthonormal basis of eigenvectors. We can let $\mathbf{v}$ be one of the basis vectors, and we can rotate $\mathbf{v}$ by $90°$ to get a second eigenbasis vector: $\mathbf{w} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

The product of the eigenvalues is the determinant, so the eigenvalue of $\mathbf{w}$ must be $(\det C)/5 = 7/5$.

5. Consider the basis $\mathcal{B}$ of $\mathbf{R}^2$ consisting of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

5(a). Find the matrix C such that $\mathbf{w} = C[\mathbf{w}]_{\mathcal{B}}$ for all vectors $\mathbf{w} \in \mathbf{R}^2$.

Solution: $C = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$.

5(b). Find a matrix M so that $[\mathbf{w}]_{\mathcal{B}} = M\mathbf{w}$ for all $\mathbf{w} \in \mathbf{R}^2$.

Solution: $M = C^{-1}$, which we find as usual:

$$\begin{bmatrix} 1 & 4 & | & 1 & 0 \\ 1 & 5 & | & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 4 & | & 1 & 0 \\ 0 & 1 & | & -1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & | & 5 & -4 \\ 0 & 1 & | & -1 & 1 \end{bmatrix}$$

Thus $M = C^{-1} = \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$.

5(c). Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ the the linear map such that $T(\mathbf{v}_1) = 5\mathbf{v}_1 + \mathbf{v}_2$ and $T(\mathbf{v}_2) = 7\mathbf{v}_1$. Find the matrix B for T in the coordinate system determined by $\mathcal{B}$. In other words, find a matrix B such that

$$[T(\mathbf{w})]_{\mathcal{B}} = B[\mathbf{w}]_{\mathcal{B}}$$

for all $\mathbf{w} \in \mathbf{R}^2$.

Solution: The first column is $[T(\mathbf{v}_1)]_{\mathcal{B}}$ or $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$. The second column is $[T(\mathbf{v}_2)]_{\mathcal{B}}$ or $\begin{bmatrix} 7 \\ 0 \end{bmatrix}$. Thus $B = \begin{bmatrix} 5 & 7 \\ 1 & 0 \end{bmatrix}$.

5(d). Let A be the matrix that represents T in standard Cartesian coordinates. Write an equation that expresses A in terms of B, C, and M:

$$A =$$

(You do not need to calculate A.)

Solution: Recall that $B = C^{-1}AC$, so (multiplying on the left by C and on the right by C^{-1}) we see that $A = CBC^{-1} = CBM$.

6. Let V be the subspace of $\mathbf{R}^3$ spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Consider the coordinate system for V determined by the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$.

6(a). Find the vector $\mathbf{w} \in \mathbf{R}^3$ whose expression in the $\mathcal{B}$-coordinate system is

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution: $\mathbf{w} = 1\mathbf{v}_1 + 2\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

6(b). Find $[\mathbf{v}]_{\mathcal{B}}$ (the expression for $\mathbf{v}$ in the $\mathcal{B}$-coordinate system) for the vector $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$.

Solution: $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, i.e., where $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

By Gaussian elimination (or just by inspection), $c_1 = 2$ and $c_3 = -3$. Thus $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

7(a). Find the equation of the tangent plane to the surface $z = x^2y + y^3$ at the point $(x, y, z) = (2, 1, 5)$.

Solution: Let $f(x, y, z) = x^2y + y^3$. Note that $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2 + 3y^2$. Thus $\frac{\partial f}{\partial x}(2, 1) = 4$ and $\frac{\partial f}{\partial y}(2, 1) = 7$. The equation of the tangent plane is

$$z = f(2, 1) + \frac{\partial f}{\partial x}(2, 1)(x - 2) + \frac{\partial f}{\partial y}(2, 1)(y - 1)$$

or

$$z = 5 + 4(x - 2) + 7(y - 1).$$

Alternate method: The surface is a level set of the function $g(x, y, z) = x^2y + y^3 - z$. Note that $\nabla g = (2xy, x^2 + 3y^2, -1)$. Thus $\nabla g(2, 1, 5) = (4, 7, -1)$ is a normal to the surface at the point $(2, 1, 5)$.

The equation for the tangent plane is thus

$$(4, 7, -1) \cdot ((x, y, z) - (2, 1, 5)) = 0$$

or

$$4(x - 2) + 7(y - 1) - (z - 5) = 0.$$

7(b). Find the matrix $DG(x, y)$ for the map G given by $G(x, y) = \begin{bmatrix} x^2y \\ y^3 \\ x \sin y \end{bmatrix}$.

Solution: The first column is $\frac{\partial G}{\partial x}$ or $\begin{bmatrix} 2xy \\ 0 \\ \sin y \end{bmatrix}$. The second column is $\frac{\partial G}{\partial y}$ or $\begin{bmatrix} x^2 \\ 3y^2 \\ x \cos y \end{bmatrix}$. Thus

$$DG(x, y) = \begin{bmatrix} 2xy & x^2 \\ 0 & 3y^2 \\ \sin y & x \cos y \end{bmatrix}.$$

8. Suppose that $F : \mathbf{R}^2 \to \mathbf{R}^2$ is a map such that $F(0, 0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and such that $DF(0, 0) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

8(a). Estimate $F(.002, .003)$.

Solution: Using the linear approximation at $(0, 0)$ gives

$$\begin{aligned} F(.002, .003) &\simeq F(0,0) + DF(0,0) \begin{bmatrix} .002 \\ .003 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} .002 \\ .003 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} .005 \\ .007 \end{bmatrix} = \begin{bmatrix} 3.005 \\ 1.007 \end{bmatrix}. \end{aligned}$$

8(b). Find a point (x, y) near $(0, 0)$ so that

$$(*) \qquad F(x, y) \simeq \begin{bmatrix} 3.004 \\ 1.007 \end{bmatrix}.$$

Solution: If we use linear approximation to estimate $F(x, y)$, then (*) becomes

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \simeq \begin{bmatrix} 3.004 \\ 1.007 \end{bmatrix},$$

so

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \simeq \begin{bmatrix} .004 \\ .007 \end{bmatrix}.$$

Solving by Gaussian elimination gives $x \simeq .003$ and $y \simeq .001$.

9(a). Suppose $\mathbf{v}$ is an eigenvector of A with eigenvalue λ. Prove that $\mathbf{v}$ is also an eigenvector of $I + A^2$, and find its eigenvalue.

Solution: $(I + A^2)\mathbf{v} = I\mathbf{v} + A^2\mathbf{v} = \mathbf{v} + A(A\mathbf{v}) = \mathbf{v} + A(\lambda\mathbf{v}) = \mathbf{v} + \lambda(A\mathbf{v}) = \mathbf{v} + \lambda^2\mathbf{v} = (1 + \lambda^2)\mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector of $I + A^2$ with eigenvalue $1 + \lambda^2$.

9(b). Suppose that A and B are similar matrices, i.e., that $B = C^{-1}AC$ for some invertible matrix C. Suppose that λ is a real number. Prove that $\lambda I - A$ and $\lambda I - B$ are also similar.

$$\begin{aligned} C^{-1}(\lambda I - A)C &= C^{-1}(\lambda I)C - C^{-1}AC \\ &= \lambda C^{-1}IC - B \\ &= \lambda C^{-1}C - B \\ &= \lambda I - B \end{aligned}$$

Thus $\lambda I - A$ and $\lambda I - B$ are similar.