MATH 51 MIDTERM 1 SOLUTIONS

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1. Find all solutions of the following system:

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & x_4 & = & 7 \\ x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & = & 12 \\ 2x_1 & + & 4x_2 & & & & 6x_4 & = & 4. \end{array}$$

Solution: See page 44 of the text.

2(a). Find a parametric equation for the plane containing the points $A = (1, 2, 3)$, $B = (4, 5, 6)$, and $C = (2, 2, 3)$.

2(b). Find the equation for the plane that passes through the point $A = (1, 2, 3)$ and that is perpendicular to the vector $\mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$. (Your answer should be an equation of the form $ax + by + cz = d$.)

Solution: Note that $\mathbf{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in the plane if and only if $\mathbf{v}$ is orthogonal to $\overrightarrow{A\mathbf{P}}$, i.e., if and only if $\mathbf{v} \cdot \overrightarrow{A\mathbf{P}} = 0$, i.e., if and only if

$$\mathbf{v} \cdot (\mathbf{P} - \mathbf{A}) = 0.$$

Thus the equation is

$$\begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = 0$$

or, equivalently,

$$\boxed{7x + 3y + 5z = 28.}$$

3(a) Suppose that Δ is an equilateral triangle in $\mathbf{R}^3$ and that the edges of Δ each have length 1. Let A, B, and C be the vertices of Δ. Find

$$(3\overrightarrow{AB}) \cdot (5\overrightarrow{AC}).$$

Solution:

$$(3\overrightarrow{AB}) \cdot (5\overrightarrow{AC}) = 15\, \overrightarrow{AB} \cdot \overrightarrow{AC} = 15\, \|\overrightarrow{AB}\|\, \|\overrightarrow{AC}\|\, \cos\theta = 15 \cdot 1 \cdot 1 \cos\left(\frac{\pi}{2}\right) = \boxed{\frac{15}{2}}.$$

3(b). Suppose $\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ are orthogonal vectors in $\mathbf{R}^4$ with $a_4 > 0$ and $b_4 > 0$.

Let $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Prove that the angle between **a** and **b** is obtuse (i.e., greater than $\pi/2$).

Solution:

$$0 = \mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 = \mathbf{a} \cdot \mathbf{b} + a_4b_4 > \mathbf{a} \cdot \mathbf{b}$$

since a_4 and b_4 are positive. Thus

$$(*) \qquad 0 > \mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

(where θ is the angle between **a** and **b**). Since $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ are nonnegative, (*) implies that $0 < \cos\theta$, which implies that the angle is obtuse. $\square$

4. Let V be the set of vectors in $\mathbf{R}^4$ that are orthogonal to the vector $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 5 \end{bmatrix}$.

Find a basis for V.

Solution: A vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is orthogonal to **a** if and only if the dot product $\mathbf{v} \cdot \mathbf{a} = 0$ which happens if and only if

$$x + 2y + 5w = 0$$

This is a system with one equation and four unknowns. It has one pivot variable, x, and three free variables. Every solution is given by

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2y - 5w \\ y \\ z \\ w \end{bmatrix}$$

$$= y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

And the basis we are looking for is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

5. Are the following three vectors in $\mathbf{R}^3$ linearly independent or linearly dependent? Show your work and explain your answer.

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ 9 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}.$$

Solution:

Let $A = [\mathbf{u}, \mathbf{v}, \mathbf{w}] = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & -2 \\ 2 & 9 & 3 \end{bmatrix}$ Careful row reduction (you should show all steps) gives $RREF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which shows that the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.

6. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & -1 \end{bmatrix}.$$

6(a). What condition(s) must $\mathbf{b}$ satisfy to be in the column space of A?

(Your answer should be one or more equations $?\,b_1 + ?\,b_2 + ?\,b_3 + ?\,b_4 = ?$.)

Solution:

Row reducing the augmented matrix

$$[A|b]$$

yields :

$$RREF([A|b]) = \begin{bmatrix} 1 & 0 & -2 & | & 3b_1 - 2b_3 \\ 0 & 1 & 2 & | & -b_1 + b_3 \\ 0 & 0 & 0 & | & -2b_1 + b_3 + b_4 \\ 0 & 0 & 0 & | & b_2 - 2b_3 \end{bmatrix}$$

So the conditions are

$$-2b_1 + b_3 + b_4 = 0$$

$$b_2 - 2b_3 = 0$$

6(b) Find a matrix M such that the column space of A is equal to the null space of M. [Hint: use your answer to part (a).]

Solution: The column space of A is given by the two equations we found in part (a). They also give the null space of the coefficient matrix

$$\begin{bmatrix} -2 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

7(a) Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly independent vectors in $\mathbf{R}^n$. Prove that the vectors $\mathbf{x}+\mathbf{y}$, $\mathbf{x}-\mathbf{y}$, and $\mathbf{x}+\mathbf{y}+\mathbf{z}$ are also linearly independent.

Solution: Suppose

(*) $$c_1(\mathbf{x}+\mathbf{y}) + c_2(\mathbf{x}-\mathbf{y}) + c_3(\mathbf{x}+\mathbf{y}+\mathbf{z}) = \mathbf{0}.$$

Then (regrouping the terms):

(**) $$(c_1+c_2+c_3)\mathbf{x} + (c_1-c_2+c_3)\mathbf{y} + c_3\mathbf{z} = \mathbf{0}.$$

Since $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly independent, the only way a linear combination of them can equal $\mathbf{0}$ is for each coefficient to be 0. Thus from (**) we see

$$c_1+c_2+c_3=0 \qquad c_1-c_2+c_3=0 \qquad c_3=0.$$

Substituting the third equation ($c_3=0$) into the first two gives

$$c_1+c_2=0 \qquad c_1-c_2=0 \qquad c_3=0.$$

Adding the first two equations gives $2c_1=0$ and therefore that $c_1=0$. Since $c_1+c_2=0$, it follows that $c_2=0$.

We have shown that if (*) holds, then $c_1=c_2=c_2=0$. Thus the vectors $\mathbf{x}+\mathbf{y}$, $\mathbf{x}-\mathbf{y}$, and $\mathbf{x}+\mathbf{y}+\mathbf{z}$ are linearly independent. □

7(b) Suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent vectors in $\mathbf{R}^n$, and that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Prove that $\mathbf{v}_4$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Solution: Since $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly dependent, we can find coefficients c_1, c_2, c_3, and c_4 such that

(*) $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

and such that

(**) the c_i's are not all 0.

Claim: $c_4 \neq 0$.

Proof of claim: Suppose $c_4=0$. Then by (*),

(***) $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent, (***) implies that $c_1=c_2=c_3=0$. Thus c_1, c_2, c_3, and c_4 would all be 0. But that contradicts (**). The contradiction proves that our supposition (namely $c_4=0$) is false. This proves the claim.

Now by (*),

$$c_4\mathbf{v}_4 = -c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - c_3\mathbf{v}_3.$$

Since $c_4 \neq 0$, this implies that

$$\mathbf{v}_4 = -\frac{c_1}{c_4}\mathbf{v}_1 - \frac{c_2}{c_4}\mathbf{v}_2 - \frac{c_3}{c_4}\mathbf{v}_3.$$

□

8. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

The reduced echelon form for A is

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(You do not need to check this.)

8(a) (3 points) Find a basis for the column space $C(A)$ of A.

Solution:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

8(b) (4 points) Find a basis for the nullspace $N(A)$ of A.

Solution:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

8(c) (3 points) Find all solutions $\mathbf{x}$ of $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$. [Hint: compare the right hand side of this equation to the columns of A.]

Solution:

We need to find a particular solution first. Notice that the two vectors on the right are the equal to the third and the fourth column vector of A. Hence

$$A\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\2\\0\\1\end{bmatrix}\qquad A\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\2\\1\end{bmatrix}\qquad \text{and hence } A\begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\2\\0\\1\end{bmatrix}+\begin{bmatrix}0\\1\\2\\1\end{bmatrix}$$

Then all solutions to the system are given by

$$\left\{\begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}+s\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}+t\begin{bmatrix}2\\0\\1\\1\\-1\end{bmatrix}: s,t\in R\right\}$$

9(a,b,c). Suppose V is a set of vectors in $\mathbf{R}^n$. What three properties must V have in order to be a linear subspace of V?

Solution: It must be closed under addition, it must be closed under scalar multiplication, and it must contain the zero vector.

9(d,e). Suppose that V is a linear subspace of $\mathbf{R}^n$ and that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are vectors in $\mathbf{R}^n$. What two properties must $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ have in order to be a basis for V?

Solution: The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ must be linearly indepedent, and their span must be V.