MIDTERM 2

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Name: Solutions

Signature: ______________________

1	10 pts	
2	10 pts	
3	~~10~~ 13 pts	
4	~~18~~ 15 pts	
5	~~15~~ 10 pts	
6	12 pts	
7	10 pts	
Total	80 pts	

Circle your TA's name

Lan
Oren
Josh
Peter
Chad
Leo
Rob
Nikola
Jian

(1) (10 point) Find the determinant of each of the following matrices. Show your work or justify your answer.

(a) $\begin{pmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ 3 & 5 & 6 & 9 \\ 2 & 0 & 3 & 3 \end{pmatrix}$

$\det(A) = -3\det\begin{pmatrix} 1 & 2 & 2 \\ 3 & 6 & 9 \\ 2 & 3 & 3 \end{pmatrix} - 5\det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix}$

$= -3\left[\det\begin{pmatrix} 6 & 9 \\ 3 & 3 \end{pmatrix} - 2\det\begin{pmatrix} 3 & 9 \\ 2 & 3 \end{pmatrix} + 2\det\begin{pmatrix} 3 & 6 \\ 2 & 3 \end{pmatrix}\right] - 5\left[2\det\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} - \det\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\right]$

$= -3\left[-9 + 18 - 6\right] - 5\left[0 + 1\right] = -3(3) - 5 = -14$

(b) $\begin{pmatrix} 1 & 2 & 3 & 5 & 10 & 0 & 3 \\ 2 & 3 & 1 & 1 & 0 & 0 & 2 \\ 3 & 5 & 4 & 6 & 10 & 0 & 5 \\ 0 & 7 & 2 & 3 & 4 & 2 & 1 \\ 4 & 1 & 8 & 9 & 10 & 6 & 7 \\ 3 & 1 & 5 & 1 & 1 & 10 & 3 \\ 2 & 0 & 0 & 3 & 4 & 1 & 8 \end{pmatrix}$ (Hint: What is row 3 in terms of rows 1 and 2?)

$\det(A) = 0$ since $A \sim \begin{pmatrix} 1 & 2 & 3 & 5 & 10 & 0 & 3 \\ 2 & 3 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 2 & 3 & 4 & 2 & 1 \\ 4 & 1 & 8 & 9 & 10 & 6 & 7 \\ 3 & 1 & 5 & 1 & 1 & 10 & 3 \\ 2 & 0 & 0 & 3 & 4 & 1 & 8 \end{pmatrix} = B$

(subtract I and II from III), But $\det(B) = 0$ since B has a row of zeros, and $\det(A) = \det(B)$ because the row operations we performed preserve determinant.

(2) ~~(4)~~ (10 points) Suppose that $V \subseteq \mathbb{R}^4$ is a subspace with basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 4 \\ 0 \end{bmatrix} \right\}.$$

(a) Use the Gram-Schmidt process and $\mathcal{B}$ to produce an orthonormal basis for V.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\text{above}\|} = \frac{\begin{bmatrix} 6 \\ 3 \\ 4 \\ 0 \end{bmatrix} - \left(\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 4 \\ 0 \end{bmatrix}\right)\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}}{\|\text{above}\|}$$

$$= \frac{\begin{bmatrix} 6 \\ 3 \\ 4 \\ 0 \end{bmatrix} - \frac{1}{9}(18)\begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}}{\|\text{above}\|} = \frac{\begin{bmatrix} 6 \\ 3 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 0 \\ 2 \end{bmatrix}}{\|\text{above}\|} = \frac{1}{5}\begin{bmatrix} 2 \\ -1 \\ 4 \\ -2 \end{bmatrix}.$$

(b) Compute the orthogonal projection of $\vec{x} = \begin{bmatrix} 9 \\ 0 \\ 25 \\ 9 \end{bmatrix}$ onto V.

$$\text{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2$$

$$= \left(\begin{bmatrix} 9 \\ 0 \\ 25 \\ 9 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right)\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \left(\begin{bmatrix} 9 \\ 0 \\ 25 \\ 9 \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 2 \\ -1 \\ 4 \\ -2 \end{bmatrix}\right)\frac{1}{5}\begin{bmatrix} 2 \\ -1 \\ 4 \\ -2 \end{bmatrix}$$

$$= \frac{1}{9}(27)\begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{25}(18 + 100 - 18)\begin{bmatrix} 2 \\ -1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 8 \\ -4 \\ 16 \\ -8 \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \\ 16 \\ -5 \end{bmatrix}.$$

(3)
~~(4)~~ (13 points) One can show that $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ is a basis for $\mathbb{R}^4$, where

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

(You don't have to prove this!)

(a) Give the matrix C which changes basis from $\mathcal{B}$ to the standard basis. That is, find C so that

$$C[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{S}},$$

where $[\vec{v}]_{\mathcal{B}}$ is $\vec{v}$ in $\mathcal{B}$-coordinates and $[\vec{v}]_{\mathcal{S}}$ is $\vec{v}$ in standard coordinates.

$$C = \begin{pmatrix} 1 & 5 & 1 & 1 \\ 2 & 6 & 1 & 0 \\ 3 & 7 & 0 & 3 \\ 4 & 8 & 0 & 0 \end{pmatrix}$$

(b) Suppose that $T : \mathbb{R}^4 \to \mathbb{R}^4$ is the linear transformation which satisfies

- $T(\vec{v_1}) = \vec{v_1} + 2\vec{v_2}$,
- $T(\vec{v_2}) = \vec{v_1} + \vec{v_3}$,
- $T(\vec{v_3}) = \vec{v_1} + \vec{v_2} - \vec{v_4}$, and
- $T(\vec{v_4}) = \vec{0}$.

Give the matrix for T in coordinates relative to the basis $\mathcal{B}$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(c) Give the matrix for T in coordinates relative to the standard basis. (You may express your answer as a product of matrices and their inverses without expanding out the products or computing the inverses).

$$\begin{pmatrix} 1 & 5 & 1 & 1 \\ 2 & 6 & 1 & 0 \\ 3 & 7 & 0 & 3 \\ 4 & 8 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 5 & 1 & 1 \\ 2 & 6 & 1 & 0 \\ 3 & 7 & 0 & 3 \\ 4 & 8 & 0 & 0 \end{pmatrix}^{-1}$$

(4) ~~4.~~ (15 points) Let

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 5 & 1 \\ 0 & -9 & -1 \end{pmatrix}.$$

(a) Find all eigenvalues of A.

$$\det(A - \lambda I_3) = \det\begin{pmatrix} 1-\lambda & 3 & 1 \\ 0 & 5-\lambda & 1 \\ 0 & -9 & -1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\det\begin{bmatrix} 5-\lambda & 1 \\ -9 & -1-\lambda \end{bmatrix} = (1-\lambda)\left[(5-\lambda)(-1-\lambda) + 9\right]$$

$$= (1-\lambda)(-5 - 5\lambda + \lambda + \lambda^2 + 9) = (1-\lambda)(\lambda^2 - 4\lambda + 4)$$

$$= (1-\lambda)(\lambda-2)^2.$$

So eigenvalues are $1, 2$.

(b) For each eigenvalue, give a basis for the corresponding eigenspace.

$$E_1 = N\begin{pmatrix} 0 & 3 & 1 \\ 0 & 4 & 1 \\ 0 & -9 & -2 \end{pmatrix}$$ This matrix has $\dim(C(\cdot)) = 2$

Since $\begin{pmatrix} 3 \\ 4 \\ -9 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ are independent, and so $\dim(N(\cdot)) = 1$

But $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is visibly in $N(\cdot)$, and so

$$E_1 = \operatorname{span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$E_2 = N\begin{pmatrix} -1 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & -9 & -3 \end{pmatrix} = N\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$, and hence

$$E_2 = \operatorname{span}\begin{pmatrix} 0 \\ -1/3 \\ 1 \end{pmatrix}.$$

(c) Is A diagonalizable? Be sure to explain your answer fully.

No. Since $\sum_\lambda \dim(E_\lambda) = 2 < 3$, A is not diagonalizable.

(5) (10 points) Suppose that $\mathcal{B} = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, and let Q be the (square) matrix

$$Q = \begin{pmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_n} \\ | & & | \end{pmatrix}.$$

(a) What is $Q^T Q$? Justify your answer.

$$Q^T Q = \begin{pmatrix} - \vec{v}_1 - \\ \vdots \\ - \vec{v}_n - \end{pmatrix} \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

Hence the ij^{th} entry of $Q^T Q$ is $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ because $\{\vec{v}_i\}$ are orthonormal. This means off-diagonal entries are 0, while all diagonal entries are 1. So

$$Q^T Q = I_n.$$

mutual orthogonality

(b) Prove that $\|Q\vec{x}\| = \|\vec{x}\|$.

Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Then $Q\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$.

Hence $\|Q\vec{x}\|^2 = Q\vec{x} \cdot Q\vec{x} = (x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) \cdot (x_1\vec{v}_1 + \cdots + x_n\vec{v}_n)$

$= (x_1 v_1) \cdot (x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) + (x_2\vec{v}_2) \cdot (\vec{x}_1 \vec{v}_n + \cdots + x_n\vec{v}_n) + \cdots + (x_n\vec{v}_n) \cdot (x_1\vec{v}_1 + \cdots + x_n\vec{v}_n)$

$= (x_1\vec{v}) \cdot (x_1\vec{v}_1) + (x_2\vec{v}_2) \cdot (x_2\vec{v}_2) + \cdots + (x_n\vec{v}_n) \cdot (x_n\vec{v}_n)$

$= x_1^2(\vec{v}_1 \cdot \vec{v}_1) + x_2^2(\vec{v}_2 \cdot \vec{v}_2) + \cdots + x_n^2(\vec{v}_n \cdot \vec{v}_n)$

$= x_1^2 + \cdots + x_n^2 = \|\vec{x}\|^2$. Take square root to finish.

each $\vec{v}_i$ has magnitude 1

Alternate proof: $\|Q\vec{x}\|^2 = (Q\vec{x}) \cdot (Q\vec{x}) = (Q\vec{x})^T Q\vec{x} = \vec{x}^T Q^T Q \vec{x} = \vec{x}^T \vec{x} = \vec{x} \cdot \vec{x} = \|\vec{x}\|^2$

(6) (12 points) Let V be a k-dimensional subspace of $\mathbb{R}^n$ for some $0 < k < n$, and let T be the linear transformation which projects vectors onto V.

(a) Prove that there exists a nontrivial vector in $V^\perp$.

$\dim(V^\perp) = n - \dim(V) = n - k > 0$ (since $k < n$).
Hence $\dim(V^\perp) \geq 1$, and so there exists nonzero $\vec{w} \in V^\perp$.

(b) Prove that 0 is an eigenvalue of T.

With $\vec{w}$ the nonzero element of $V^\perp$, we know

$\text{proj}_V(\vec{w}) = \vec{0}$, or (more suggestively)

$$\cancel{\text{proj}}\ T(\vec{w}) = 0\vec{w}.$$

This means 0 is an eigenvalue.

(c) Prove that 1 is an eigenvalue of T.

Since $\dim(V) \geq 1$, there exists a nonzero vector $\vec{v} \in V$. For this $\vec{v}$ we have $\text{proj}_V(\vec{v}) = \vec{v}$, or (more suggestively)

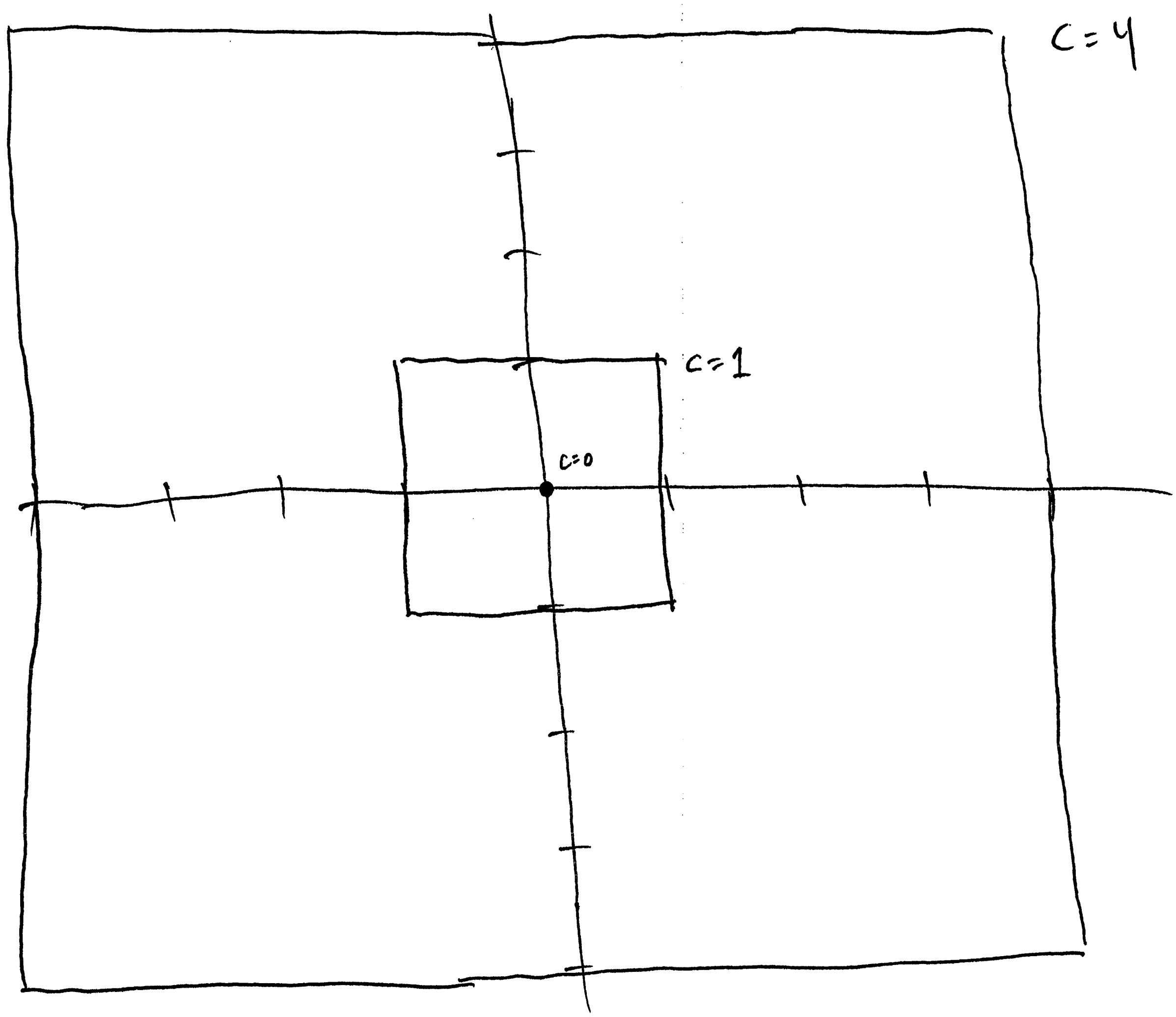
$$T(\vec{v}) = 1\vec{v}.$$

Hence 1 is an eigenvalue of projection.

(7) (10 points)
(a) Let

$$f(x,y) = \begin{cases} |y|, & \text{if } |x| \le |y| \\ |x|, & \text{if } |y| < |x|. \end{cases}$$

Carefully draw (and label) the level curves $f(x,y) = 0$, $f(x,y) = 1$ and $f(x,y) = 4$.



(b) Determine the value of the constant c so that

$$g(x,y) = \begin{cases} \dfrac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous. Be sure to justify your answer.

Notice that

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+xy^2+2x^2+2y^2}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x+2)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} (x+2) = 2.$$

Hence if we let $g(0,0) = 2$, then

$$g(0,0) = \lim_{(x,y)\to(0,0)} g(x,y)$$

and so g will be continuous at $(0,0)$.

Since at any point $(x,y) \neq (0,0)$, the function $g(x,y)$ is a quotient of polynomials (where the denominator is nonzero), the function $g(x,y)$ is automatically continuous away from $(0,0)$.

With these two statements together, $g(x,y)$ is continuous everywhere.