MIDTERM 1 - SOLUTIONS

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Name: ________________________________

Signature: ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	15 pts	
2	10 pts	
3	15 pts	
4	15 pts	
5	15 pts	
6	10 pts	
7	15 pts	
Total	95 pts	

(1) (15 points) Complete each of the following sentences.

(a) A collection of vectors $\overrightarrow{v_1}, \cdots, \overrightarrow{v_m}$ is defined to be linearly dependent if

one of the vectors in the collection is a linear combination of the other vectors in the collection. (You could have also said "if there exists scalars $c_1, \cdots, c_m$—not all zero—so that

$$c_1\overrightarrow{v_1} + \cdots + c_m\overrightarrow{v_m} = \overrightarrow{0}.)$$

(b) The dimension of a subspace W is defined to be

the number of elements in a basis for W.

(c) The book lists 4 properties a matrix must have to be in reduced row echelon form. Two of these properties are

(i) the leading entry in every row (i.e., 'pivot') is 1 and
(ii) if a column contains a pivot, then every other entry in the column is 0.

You could have also used

(iii) if a row has a pivot, any pivots in rows below are further to the right, or
(iv) if every entry in a row is 0, then all rows below also have 0 for every entry.

(2) (10 points) Compute the reduced row echelon form of the matrix

$$\begin{pmatrix} 1 & 3 & 4 & 0 \\ 1 & -1 & 0 & 1 \\ 2 & 3 & 5 & 3 \\ -2 & -1 & -3 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 3 & 4 & 0 \\ 1 & -1 & 0 & 1 \\ 2 & 3 & 5 & 3 \\ -2 & -1 & -3 & 1 \end{pmatrix} \begin{matrix} \\ -I \\ -2I \\ +2I \end{matrix} \sim \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & -4 & -4 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & 5 & 5 & 1 \end{pmatrix} \begin{matrix} \\ swap \\ swap \\ \\ \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & -4 & -4 & 1 \\ 0 & 5 & 5 & 1 \end{pmatrix} \begin{matrix} \\ -\frac{1}{3} \\ \\ \\ \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -4 & -4 & 1 \\ 0 & 5 & 5 & 1 \end{pmatrix} \begin{matrix} -3II \\ \\ +4II \\ -5II \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{matrix} \\ \\ -\frac{1}{3} \\ \\ \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{matrix} -3III \\ +III \\ \\ -6III \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(3) (15 points) A matrix B and its reduced row echelon form are given below:

$$B = \begin{pmatrix} 1 & 2 & 4 & 11 & 7 \\ 2 & 4 & 5 & 13 & 8 \\ 3 & 6 & 6 & 15 & 10 \end{pmatrix} \quad \text{and} \quad \texttt{rref}(B) = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Give a basis for the null space of B. You do not need to prove that your collection is a basis.

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

(b) Give a basis for the column space of B. You do not need to prove that your collection is a basis.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix}$$

(c) Find a non-zero vector orthogonal to all three of the vectors $\begin{pmatrix} 1 \\ 2 \\ 4 \\ 11 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 5 \\ 13 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 6 \\ 15 \\ 10 \end{pmatrix}$.

Any (non-zero) vector in the nullspace will do, so you could take any linear combination of the vectors you found in part (a).

(4) (15 points) Determine whether each statement is true or false and circle your answer. In all these statements, A is an arbitrary $m \times n$ matrix and $R = \texttt{rref}(A)$. No justification is necessary for your answers.

T (a) $N(A) = N(R)$.

F (b) $C(A) = C(R)$.

T (c) The dimensions of $C(A)$ and $C(R)$ are the same.

F (d) If the equation $A\overrightarrow{x} = \overrightarrow{0}$ has infinitely many solutions, then for any $\overrightarrow{b} \in \mathbb{R}^m$ the system $A\overrightarrow{x} = \overrightarrow{b}$ has infinitely many solutions.

T (e) If $n > m$, it is possible for $\dim(C(A)) = m$.

F (f) If $n > m$, it is possible for $\dim(C(A)) = n$.

F (g) If $n > m$, it is possible for $\dim(N(A)) = 0$.

(5) (15 points) Let P_1 and P_2 be two planes passing through the point $(1,1,1)$ such that $\overrightarrow{n}_1$ is normal to P_1 and $\overrightarrow{n}_2$ is normal to P_2, where

$$\overrightarrow{n}_1 = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \quad \text{and} \quad \overrightarrow{n}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

(a) Find the parametric equation of the line ℓ which is the intersection of P_1 and P_2.

The equations corresponding to P_1 and P_2 are

$$\left\{ \begin{array}{ll} 3x + y & +5z = 9 \\ x & +2z = 3 \end{array} \right\}.$$

The line which is the intersection of these two planes is given by the solution to the system of equations, and so we attempt to solve this system. The corresponding matrix has reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \end{array} \right),$$

and so the intersection of P_1 and P_2 is the line

$$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Prove that P_1 *is not* a subspace. (Hint: To answer this question you should find a property of subspaces that P_1 does not have. Be sure you give full justification for any claims you make.)

Subspaces should contain $\overrightarrow{0}$, and so we can show that P_1 is not a subspace by showing $\overrightarrow{0} \notin P_1$. For this, it is enough to show that $\overrightarrow{0}$ is not a solution to the equation which defines P_1: $3x + y + 5z = 9$. But clearly $x = y = z = 0$ is not a solution to this equation, and so $\overrightarrow{0} \notin P_1$.

(6) (10 points) Suppose that $\overrightarrow{x}$ and $\overrightarrow{y}$ are two vectors in $\mathbb{R}^n$ with equal magnitude. Prove that the vectors $\overrightarrow{x} + \overrightarrow{y}$ and $\overrightarrow{x} - \overrightarrow{y}$ are orthogonal.

To show that the desired vectors are orthogonal, we shall take their dot product and show it is 0. We compute:

$$\begin{aligned}(\overrightarrow{x} + \overrightarrow{y}) \cdot (\overrightarrow{x} - \overrightarrow{y}) &= \overrightarrow{x} \cdot \overrightarrow{x} + \overrightarrow{x} \cdot \overrightarrow{y} - \overrightarrow{y} \cdot \overrightarrow{x} - \overrightarrow{y} \cdot \overrightarrow{y} \\ &= \overrightarrow{x} \cdot \overrightarrow{x} - \overrightarrow{y} \cdot \overrightarrow{y} \\ &= \|\overrightarrow{x}\|^2 - \|\overrightarrow{y}\|^2 \\ &= 0,\end{aligned}$$

where here the first equality holds by distributivity, the second equality holds by commutativity, the third equality holds by the identity $\overrightarrow{x} \cdot \overrightarrow{x} = \|\overrightarrow{x}\|^2$, and the last equality holds by assumption $\|\overrightarrow{x}\| = \|\overrightarrow{y}\|$ (and so $\|\overrightarrow{x}\|^2 = \|\overrightarrow{y}\|^2$).

(7) (15 points) Consider a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ which satisfies

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$.

Our goal will be to write $\overrightarrow{e_1}$ as a linear combination of the input vectors above. Once we have this expression, we will be able to write $T(\overrightarrow{e_1})$ as a linear combination of the output vectors. Now to write $\overrightarrow{e_1}$ as a linear combination, we notice that

$$\texttt{rref}\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right).$$

This means that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, using the linearity of T, we have

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}.$$

(b) Suppose you are told that a linear transformation S satisfies

$$S\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ S\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\text{ and }S\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}5\\10\\15\\-5\end{bmatrix}.$$

What is the matrix A that satifies $A\overrightarrow{x}=S(\overrightarrow{x})$ for all $\overrightarrow{x}\in\mathbb{R}^3$?

The matrix for S should just be the matrix whose ith column is $S(\overrightarrow{e_i})$. Since we are given $S(\overrightarrow{e_1}), S(\overrightarrow{e_2})$ and $S(\overrightarrow{e_3})$, we know that

$$A=\begin{pmatrix}0&1&5\\0&0&10\\0&1&15\\0&0&-5\end{pmatrix}.$$