FINAL EXAM

- This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Name: ______________________________

Signature: ______________________________

1	10 pts		**9**	8 pts	
2	8 pts		**10**	5 pts	
3	5 pts		**11**	5 pts	
4	7 pts		**12**	12 pts	
5	10 pts		**13**	10 pts	
6	17 pts		**14**	10 pts	
7	10 pts		**15**	10 pts	
8	15 pts		**Total**	142 pts	

Circle your TA's name								
Lan	Oren	Josh	Peter	Chad	Leo	Rob	Nikola	Jian

(1) (10 points) Find bases of the null space and the column space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{pmatrix}.$$

(2) (8 points) What condition(s) must b_1, b_2, b_3 and b_4 satisfy so that the following system has a solution?

$$\begin{aligned} x - 3y &= b_1 \\ 3x + y &= b_2 \\ x + 7y &= b_3 \\ 2x + 4y &= b_4 \end{aligned}$$

(3) (5 points) Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be vectors in $\mathbb{R}^n$ whose magnitudes are $1, 2,$ and 3 respectively. Suppose that $\vec{x}$ is parallel to (and in the same direction as) $\vec{y}$, and $\vec{x}$ is perpendicular to $\vec{z}$. Find the constant(s) c such that $\vec{x} + \vec{y} + \vec{z}$ and $\vec{x} + c\vec{y} + \vec{z}$ are perpendicular.

(4) (7 points) A matrix A and its reduced row echelon form are shown below:

$$A = \begin{pmatrix} 1 & ? & 5 & 9 \\ 2 & ? & 6 & 10 \\ 3 & ? & 7 & 11 \\ 4 & ? & 8 & 13 \end{pmatrix} \quad \text{and} \quad \texttt{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

What is the second column of A?

(5) (10 points) A box containing pennies, nickels and dimes contains 13 coins altogether, with a total value of 83 cents. How many coins of each type are in the box?

(6) (17 points) Let

$$V = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

(a) Show that $\vec{v}_1$ and $\vec{v}_2$ belong to the orthogonal complement $V^\perp$ of V.

(b) Is $\{\vec{v}_1, \vec{v}_2\}$ a basis of $V^\perp$? Explain why or why not.

(c) Find an orthonormal basis of $V^\perp$.

(d) Find the orthogonal projection of u on V.

(7) (10 points) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be projection onto the plane P that passes through $\overrightarrow{0}$ and is orthogonal to the line spanned by $\begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix}$.

(a) Find an eigenbasis for T.

(b) Write down a matrix in standard coordinates which represents T. You can express your matrix as a product of matrices and inverses of matrices.

(8) (15 points) Globo-tech Marketing monitors the dollars spent each year by its customers on apples and oranges. With $a(k)$ representing the number of dollars spent (in millions) on apples in year k, and $o(k)$ the number of dollars spent (in millions) on oranges in year k, they determine that

$$\begin{aligned} a(k+1) &= \tfrac{2}{10}a(k) + \tfrac{4}{10}o(k) \\ o(k+1) &= \tfrac{8}{10}a(k) + \tfrac{6}{10}o(k) \end{aligned}$$

We shall write $\vec{v}_k = \begin{bmatrix} a(k) \\ o(k) \end{bmatrix}$.

(a) Find a matrix A so that $A\vec{v}_k = \vec{v}_{k+1}$. Notice that this will imply $A^k\vec{v}_0 = \vec{v}_k$.

(b) Find the eigenvalues of A, and for each eigenvalue find a basis for the corresponding eigenspace.

(c) Express $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as a linear combination of the eigenvectors you just computed.

(d) Suppose that $\overrightarrow{v}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Using your answers from above, what is a good estimate for the number of dollars (in millions) spent on apples in year 100? What about dollars (in millions) spent on oranges in year 100?

(9) (8 points) Show that if A is an $n \times n$ matrix then there exist scalars $c_0, \cdots, c_n$—not all zero—so that

$$\det(c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_n A^n) = 0.$$

(Hint: For a vector $\overrightarrow{v}$, what can you say about linear dependence of the collection $\overrightarrow{v}, A\overrightarrow{v}, \cdots, A^n\overrightarrow{v}$? Why might this help you?)

(10) (5 points) Does there exist a constant c such that

$$f(x,y) = \begin{cases} \frac{(x+y)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous? Why or why not?

(11) (5 points) Let S be the surface in $\mathbb{R}^3$ defined by

$$x^2 + \frac{y^2}{4} - z^2 = 1.$$

What is the tangent plane to this surface at the point $(1, 2, 1)$?

(12) (12 points) Consider the function $f(x, y) = x^2/y^4$.

(a) Carefully draw the level curve passing through the point $(1, -1)$. On this graph, draw the gradient of the function f at $(1, -1)$.

(b) Compute the directional derivative of f at the point $(1, -1)$ in the direction $\overrightarrow{u} = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$.

(c) Suppose that $f(x, y)$ gives the height of a mountain above (x, y), and suppose further that you are stuck on the mountain at position $(1, -1, f(1, -1))$. In what direction $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$ should you take your first step if you want to descend the mountain as quickly as possible?

(13) (10 points) Consider the function

$$f(x, y, z) = \sqrt{\ln\left(e^{2x} y z^3\right)}$$

(a) Write down the first order Taylor polynomial centered at the point $(2, 1, 1)$.

(b) Find the approximate value of the number $\sqrt{\ln(e^{4.01}(.98).(1.03)^3)}$.

(14) (10 points) Find all critical points of the function $2x^3 + 6xy + 3y^2$ and describe their nature.

(15) (10 points) Use calculus to find the point on the circle $(x-1)^2 + (y-2)^2 = 1$ which is nearest to the origin.