Math 51 Exam 1 — April 24, 2007

Name : ____

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- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- **You have 90 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	10 pts	
2	10 pts	
3	10 pts	
4	20 pts	
5	14 pts	
6	16 pts	
7	20 pts	
Total	100 pts	

1. (10 points) Complete each of the following sentences.

   (a) A collection of vectors $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$ is defined to be linearly independent if

   (b) A basis for a subspace V is defined to be

2. (10 points) Let Q be the set of all vectors in $\mathbb{R}^3$ that are orthogonal to $\overrightarrow{w} = (1, 3, -1)$.

   (a) Find a parametrization for the set Q. (Hint: Q forms a plane in $\mathbb{R}^3$.)

   (b) Suppose P is a plane in $\mathbb{R}^3$, parallel to Q, such that P passes through the point $(2, 0, -1)$. Find an equation for P, written in the form $ax + by + cz = d$.

3. (10 points) Compute, showing all steps, the reduced row echelon form of the matrix

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ -1 & 0 & -2 & 0 & -1 \\ 2 & 2 & 2 & 3 & 7 \\ -2 & 2 & -6 & 0 & 6 \end{bmatrix}.$$

4. (20 points) A matrix A and its reduced row echelon form are given:

$$A = \begin{bmatrix} 2 & 4 & 6 & 14 \\ 1 & 2 & 3 & 7 \\ 3 & 5 & 7 & 16 \end{bmatrix}, \qquad \texttt{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Give a basis for the null space of A. You do not need to prove that your collection is a basis.

(b) Give a basis for the column space of A. You do not need to prove that your collection is a basis.

For easy reference: $A = \begin{bmatrix} 2 & 4 & 6 & 14 \\ 1 & 2 & 3 & 7 \\ 3 & 5 & 7 & 16 \end{bmatrix}$, $\texttt{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) Give a specific vector $\overrightarrow{b} \in \mathbb{R}^3$ such that the equation $A\overrightarrow{x} = \overrightarrow{b}$ has infinitely many solutions (and give the solution set in parametric form), or state why such a $\overrightarrow{b}$ does not exist.

(d) Give a specific vector $\overrightarrow{b} \in \mathbb{R}^3$ such that the equation $A\overrightarrow{x} = \overrightarrow{b}$ has exactly one solution (and give this solution), or state why such a $\overrightarrow{b}$ does not exist.

(e) Give a specific vector $\overrightarrow{b} \in \mathbb{R}^3$ such that the equation $A\overrightarrow{x} = \overrightarrow{b}$ has no solutions (and explain why), or state why such a $\overrightarrow{b}$ does not exist.

5. (14 points) Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

Note: In the statements below, A is an arbitrary $m \times n$ matrix.

T **F** The dimension of a subspace V equals the number of vectors contained in V.

T **F** The cross-product of any two collinear vectors in $\mathbb{R}^3$ is the zero vector.

T **F** If $\overrightarrow{v}$ and $\overrightarrow{w}$ are orthogonal vectors in $\mathbb{R}^n$, then $A\overrightarrow{v}$ and $A\overrightarrow{w}$ must be orthogonal vectors in $\mathbb{R}^m$.

T **F** For any $\overrightarrow{b} \in \mathbb{R}^m$ for which there are infinitely many solutions to the equation $A\overrightarrow{x} = \overrightarrow{b}$, this set of solutions is a subspace of $\mathbb{R}^n$.

T **F** If $m > n$, it is possible for A to have rank m.

T **F** If $m > n$, it is possible for A to have rank n.

T **F** If $m > n$, it is possible for $\dim N(A) = n$.

6. (16 points)

(a) Suppose $G : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$G\left(\begin{bmatrix}2\\1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\-1\end{bmatrix},\ G\left(\begin{bmatrix}3\\1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\7\end{bmatrix},\ \text{and } G\left(\begin{bmatrix}3\\0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}.$$

Compute $G\left(\begin{bmatrix}1\\2\\0\end{bmatrix}\right)$.

(b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates vectors $\pi/2$ radians counterclockwise. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects vectors across the y-axis. Find the matrices representing the transformations S, T, and $S \circ T$. (By the matrix representing a linear transformation F, we mean the matrix A such that $F(\overrightarrow{x}) = A\overrightarrow{x}$ for all $\overrightarrow{x}$ in $\mathbb{R}^2$.)

S:

T:

$S \circ T$:

7. (20 points) Complete the following questions, showing your reasoning.

(a) Let $V = \text{span}\left(\begin{bmatrix} 8 \\ 7 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$. Find a matrix A such that $C(A) = V$.

(b) Let W be the set of all vectors in $\mathbb{R}^3$ which are orthogonal to both $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 3 \\ -1 \end{bmatrix}$.

Find a matrix A such that $N(A) = W$.

(c) Let $R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} 2x + y + z = 0 \text{ and} \\ x + y - z = 1 \end{matrix} \right\}$.

Find a matrix A such that R is the set of all $\overrightarrow{v} \in \mathbb{R}^3$ satisfying $A\overrightarrow{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

(d) Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x - y + 2z = 0 \text{ and} \\ 2x - y + 3z = 0 \end{matrix} \right\}$. Find a matrix A such that $C(A) = S$.