Math 51 Final Exam — June 8, 2007

Name : ____

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- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	14 pts		**9**	8 pts	
2	12 pts		**10**	10 pts	
3	8 pts		**11**	12 pts	
4	12 pts		**12**	10 pts	
5	16 pts		**13**	16 pts	
6	12 pts		**14**	14 pts	
7	12 pts		**15**	12 pts	
8	20 pts		**16**	12 pts	
			Total	200 pts	

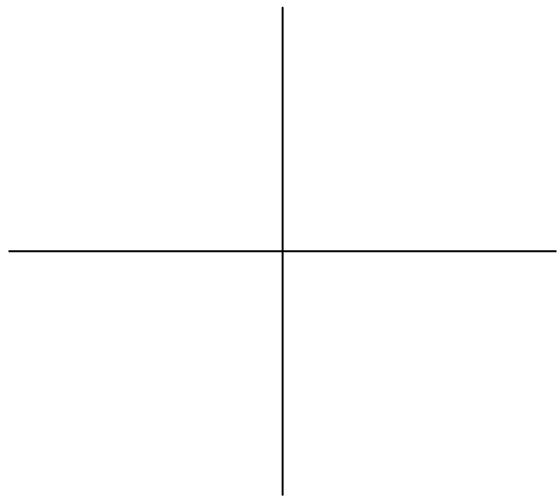
1. (14 points) Let $f(x, y) = \frac{1}{2}x^2 + \frac{3}{2}y^2 - xy^3$.

   (a) Find all the critical points of f. For each, specify if it is a local maximum, a local minimum, or a saddle point, and briefly show how you know.

   (b) Write the quadratic approximation (that is, the degree-2 Taylor polynomial) for f at the point $(x, y) = (1, 1)$.

2. (12 points) Consider the function $f(x, y) = \sqrt{50 - x^2 - y^2}$.

(a) Find an equation that defines the level set of f through the point $(x, y) = (3, 4)$. Sketch and label the curve and point on the axes below. (Be sure to include the scales on your axes.)



(b) Calculate ∇f, the gradient of f, at the point $(x, y) = (3, 4)$ and indicate it on your diagram above.

(c) Calculate the directional derivative of f at the point $(3, 4)$ in the direction of the vector $(2, -1)$.

3. (8 points) Suppose S is the surface in $\mathbb{R}^3$ given by the equation $xy + yz + xz = 1$.

   (a) Find the equation of the tangent plane to S at the point $(x, y, z) = (-1, 2, 3)$.

   (b) Use linear approximation to estimate the value of z for the point on S where $x = -1.01$ and $y = 2.02$.

4. (12 points)

(a) Assume $h(x, y) = g(x^2 + y^2)$, where g is a function of one variable. Find $h_x(1, 2) + h_y(1, 2)$, given that $g'(5) = 3$.

(b) Suppose $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

- $\mathbf{F}(1, 1) = (-2, -3)$, $\quad \mathbf{F}(-2, -3) = (0, 2)$, $\quad \mathbf{F}(0, 2) = (1, 1)$,
- $D\mathbf{F}(1, 1) = \begin{bmatrix} 0 & 4 \\ -1 & 1 \end{bmatrix}$, $\quad D\mathbf{F}(-2, -3) = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$, $\quad D\mathbf{F}(0, 2) = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$.

Find $D(\mathbf{F} \circ \mathbf{F})(1, 1)$.

5. (16 points) It's a little-known fact that Silicon Valley got its name from the rich underground deposits of this metal throughout Santa Clara County. Recently, Hewlett Packard approached the Stanford trustees with shocking news: silicon even lies beneath the Stanford Oval, which is the region

$$R = \{(x, y) \mid x^2 + 4y^2 \leq 100\},$$

and HP would happily dig some up for the trustees, taking a cut for themselves.

Under fire from student protest groups, the University decides to allow only two dig sites: one for HP to keep, and one for the trustees to sell to the highest bidder. The Geology Department informs the University that the value V of the silicon obtained from a dig site with coordinates (x, y) will be given by the formula $V = 200 + 18y - x^2 - y^2$.

(a) What are the coordinates (x, y) in the region R for the most valuable dig site (for the trustees) and the least valuable site (for HP), and what are the values at these points?

(b) The trustees also want to know if they could do even better if they're able to dig outside the Oval. Determine whether or not there is a dig site with even greater V, and if so, the location and value of the maximum.

6. (12 points) The lines L_1 and L_2 in $\mathbb{R}^3$ are given by the parametric equations

$$L_1 : \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+2s \\ -1+s \\ s \end{bmatrix} \;\middle|\; s \in \mathbb{R} \right\}, \qquad L_2 : \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

(a) Show that L_1 and L_2 do not intersect. (Hint: show that there is no pair (s, t) of parameters satisfying the three component equations.)

(b) Find the shortest distance between two points P and Q, where P lies on L_1 and Q lies on L_2.

7. (12 points) Find the point in $\mathbb{R}^3$ closest to the origin and lying on both the planes

$$\begin{aligned} x - 2y - 2z &= 1 \quad \text{and} \\ 2x - y + 2z &= 2. \end{aligned}$$

8. (20 points) Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

T **F** If $\overrightarrow{u}$ and $\overrightarrow{v}$ are orthogonal vectors in $\mathbb{R}^3$, then $(\overrightarrow{u} \times \overrightarrow{v}) \times \overrightarrow{u}$ is a scalar multiple of $\overrightarrow{v}$.

T **F** If $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, and $\overrightarrow{e_3}$ are the standard basis vectors of $\mathbb{R}^3$, then $\overrightarrow{e_1} \cdot (\overrightarrow{e_2} \times \overrightarrow{e_3}) = \overrightarrow{e_3} \cdot (\overrightarrow{e_2} \times \overrightarrow{e_1})$.

T **F** The identity $|\overrightarrow{v} \cdot \overrightarrow{w}|^2 + ||\overrightarrow{v} \times \overrightarrow{w}||^2 = ||\overrightarrow{v}||^2 ||\overrightarrow{w}||^2$ holds for all vectors $\overrightarrow{v}, \overrightarrow{w}$ in $\mathbb{R}^3$.

T **F** There are no functions $f(x, y)$ for which every point on the unit circle is a critical point.

T **F** If $f(x, y) = \sin(\cos(y + x^{14}) + \cos x)$, then $f_{xyxyx} = f_{yyxxx}$.

T **F** The function $f(x, y) = -x^{2010} - y^{2010}$ has a critical point at $(0, 0)$, which is a local minimum.

T **F** The maximum of $f(x, y)$ under the constraint $g(x, y) = 0$ is the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

T **F** An absolute maximum (x_0, y_0) of $f(x, y)$ is also an absolute maximum of $f(x, y)$ when constrained to a curve $g(x, y) = c$ that goes through the point (x_0, y_0).

T **F** Suppose $h(x, y, z)$ has a critical point at (x_0, y_0, z_0), where the six "mixed" second-order partial derivatives of h are zero, and the other three second-order partials have $h_{xx} < 0$, $h_{yy} > 0$, and $h_{zz} < 0$; then h must have a local maximum at (x_0, y_0, z_0).

T **F** The limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ does not exist.

9. (8 points) Suppose that $\overrightarrow{v}$ and $\overrightarrow{w}$ are vectors in $\mathbb{R}^n$ with $||\overrightarrow{v}|| = 2$ and $||\overrightarrow{w}|| = 3$. In each case below, $||2\overrightarrow{v} - \overrightarrow{w}||$ is given. Decide whether $\overrightarrow{v}$ and $\overrightarrow{w}$ are perpendicular or not, or whether there is not enough information to decide, and give your reasoning:

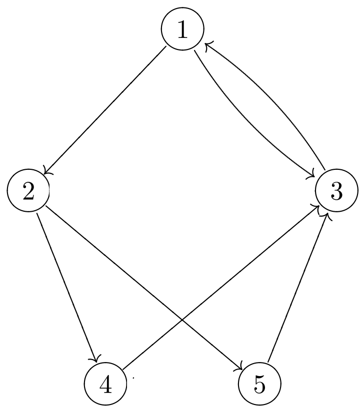
(a) $||2\overrightarrow{v} - \overrightarrow{w}|| = 7$

(b) $||2\overrightarrow{v} - \overrightarrow{w}|| = 5$

10. (10 points) Recall that Google's PageRank process begins by constructing the "linking matrix" A, whose ij entry is

$$a_{ij} = \begin{cases} 1 & \text{if page } j \text{ has a link to page } i\text{, or if page } j \text{ is a dead end} \\ 0 & \text{if not} \end{cases}$$

(a) For the universe of five web pages below, write out the linking matrix A.



$A =$

(b) Let $\vec{u}$ be the vector in $\mathbb{R}^5$ all of whose entries are 1. What is the meaning of the vector $A\vec{u}$, phrased in terms of web pages? Be as precise as possible. (You don't need to compute it for this case. You can ignore the issue of dead-end pages, for simplicity's sake.)

(c) What is the meaning of $A^T\vec{u}$? Be as precise as possible. (Again, no need to compute it; you can also ignore dead-end pages.)

11. (12 points) Use any method you like to solve the following problems, but be sure show your steps and briefly give reasoning.

(a) Determine whether the matrix below is invertible:

$$\begin{bmatrix} 1 & \frac{1}{7} & 0 & 3 & 2 \\ 0 & -1 & 0 & 0 & 0 \\ \pi & -1 & 2 & 6 & \frac{3}{5} \\ 5 & 2 & 0 & 1 & -2 \\ 2 & 3 & 0 & 2 & -1 \end{bmatrix}$$

(b) Find a nonzero vector orthogonal to the three vectors

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 7 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \\ 12 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 6 \\ 5 \end{bmatrix}.$$

12. (10 points)

(a) Suppose S is the linear transformation given by multiplication by the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$. Find two different vectors $\overrightarrow{u}, \overrightarrow{v}$ satisfying $S(\overrightarrow{u}) = S(\overrightarrow{v})$.

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and suppose there exist two different vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^n$, for which $T(\overrightarrow{u}) = T(\overrightarrow{v})$. Can T be an *onto* transformation? Why or why not?

13. (16 points) Consider the linear system $A\overrightarrow{x} = \overrightarrow{b}$, where $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(a) Determine conditions on the entries of $\overrightarrow{b}$ so that the system $A\overrightarrow{x} = \overrightarrow{b}$ has a solution. (Give your answer in the form of one or more linear equations involving only the entries of $\overrightarrow{b}$.)

(b) Now let $\overrightarrow{b} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$; in this case, the system $A\overrightarrow{x} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ has no solution. Instead, find its "least-squares" (approximate) solution $\overrightarrow{x^*} = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$.

(c) Still with $\overrightarrow{b} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ as in part (b), find a formula for a function $f(x_1, x_2)$ which has a single critical point that is located at the point $\overrightarrow{x^*}$ of part (b). You don't have to prove that f actually has a critical point at $\overrightarrow{x^*}$. (Hint: what's "least" about "least-squares"?)

14. (14 points) Suppose P is the plane in $\mathbb{R}^3$ that passes through the origin and is orthogonal to the line spanned by $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. We define two linear transformations on $\mathbb{R}^3$:

- $S : \mathbb{R}^3 \to \mathbb{R}^3$ is projection onto the plane P, and
- $T : \mathbb{R}^3 \to \mathbb{R}^3$ is *reflection* through the plane P.

(a) Find an eigenbasis for S, and write the matrix for S *with respect to this basis.*

(b) Write down the matrix for T *with respect to the standard basis.* (If you wish, you can leave your answer to (b) expressed in terms of products of matrices and inverses of matrices.)

15. (12 points) Let M be the matrix $\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$.

(a) Find the eigenvalues of M.

(b) For each eigenvalue λ of M, find a *unit* vector $\overrightarrow{u}$ in $\mathbb{R}^2$ satisfying $M\overrightarrow{u} = \lambda\overrightarrow{u}$.

(c) Find an *orthogonal* matrix C such that $C^{-1}MC$ is a diagonal matrix.

16. (12 points)

(a) Is the matrix

$$M = \begin{bmatrix} -1 & 1 & 2 & 8 & 19 & 1 & 1 \\ 0 & 1 & 3 & 2 & -11 & 3 & 0 \\ 0 & 0 & 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 3 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 17 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

diagonalizable? Explain how you know, and find $\det M$.

(b) The characteristic polynomial of the matrix

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -6 & -2 & 3 \\ -4 & -2 & 3 \end{bmatrix}$$

is $(\lambda - 1)^2(\lambda + 2)$. (You do *not* need to verify this.) Determine whether A is diagonalizable.