Problem 1. (10 pts.) a) Write the definition of when λ is an eigenvalue of A.

There is a non-zero vector $\vec{v}$ such that

$$A \cdot \vec{v} = \lambda \cdot \vec{v}$$

b) Let A be the following 3×3 matrix:

$$A = \begin{bmatrix} \star & 2 & \star \\ 1 & -3 & 0 \\ \star & -1 & \star \end{bmatrix}$$

Where $\star$ denotes the entries of A that are unknown.

Assuming that the vector $\vec{v} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$ is an eigenvector of A, find the corresponding eigenvalue.

$$A\vec{v} = \begin{bmatrix} * \\ 4-12 \\ * \end{bmatrix} = \lambda \cdot \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

So: $-8 = \lambda \cdot 4$

i.e. $\boxed{\lambda = -2}$

Problem 2. (10 pts.) Let A be the following matrix:

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 4 & 1 \\ -2 & -2 & 1 \end{bmatrix}$$

Find a basis of the eigenspace corresponding to the eigenvalue $\lambda = 2$.

$$A - 2I = \begin{bmatrix} -2 & -2 & -1 \\ 2 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} +2 & +2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

taking e.g. x_2 and x_3 as parameters we get:

$$N(A - 2I) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Problem 3. (10 pts.) Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right]$$

So: $$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -1 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

b) For the same matrix A find all solutions to the equation

$$A^{-1}\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\vec{x} = A \cdot A^{-1}\vec{x} = A \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

Problem 4. (10 pts.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ -x + y + z \\ -2x - y + z \end{bmatrix}$$

a) Find the derivative DT of the map T.

$$DT = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ -2 & -1 & 1 \end{bmatrix}$$

b) Find the inverse of T or show that this inverse does not exist.

$$\det \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ -2 & -1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & \textcircled{1} \\ -1 & -2 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = 0$$

so the inverse doesn't exist.

Problem 5. (10 pts.) A 3×3 matrix M has two linearly independent eigenvectors corresponding to eigenvalue $\lambda = 4$ and one eigenvector corresponding to eigenvalue $\lambda = -1$.

a) Is M diagonalizable? Explain your answer.

eigenvectors corresponding to e-value 4 are independent on e-vectors corresponding to e-value -1, so A has a basis consisting of e-vectors, thus is diagonalizable

b) Find $\det M$.

$\det M =$ product of e-values (with multiplicity)

$= (4)^2 \cdot (-1) = -16.$

Problem 6. Find an equation of the tangent plane to the graph of the function

$$f(x, y) = x^3y^2 - y^2x + x^2 + 1$$

at the point $(1, 1, 2)$.

Normal vector: $\begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} = \begin{bmatrix} -3x^2y^2 + y^2 - 2x \\ -2yx^3 + 2yx \\ 1 \end{bmatrix}$

at $(1,1,2)$ $\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

equation of the plane:

$$\boxed{-4(x-1) + \cancel{0\cdot(y-1)} + (z-2) = 0}$$

Problem 7. (10 pts.) Let $Q(x, y) = x^2 - 2axy + y^2$

a) For what values of the parameter a is the quadratic form Q positive definite?

Solution I: by completion of the square:

$$Q(x,y) = (x-ay)^2 + (1-a^2)\cdot y^2$$

So: $1-a^2 > 0$ i.e $1 > a^2$ i.e. $-1 < a < 1$

Solution II Q has a corresponding matrix $\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}$ with characteristic polynomial $(1-\lambda)^2 - a^2$ whose roots are $\lambda = 1 \pm a$. For Q to be positive definite $1+a > 0$ and $1-a > 0$

so: $\boxed{1 > a > -1}$

b) For what values of the constant a does the function $Q(x, y)$ satisfy the following expression:

$$\frac{\partial^2 Q}{\partial x \partial y} = \frac{\partial^2 Q}{\partial x^2}$$

$$\frac{\partial^2 Q}{\partial x \partial y} = \frac{\partial}{\partial x}(-2ax + 2y) = -2a$$

$$\frac{\partial^2 Q}{\partial x^2} = 2$$

so $\boxed{a = -1}$

Problem 8. (10 pts.) Let $\vec{x} : \mathbb{R} \to \mathbb{R}^2$ be defined by:

$$\vec{x}(t) = \begin{bmatrix} 2\cos t \\ \sin t \end{bmatrix}$$

and $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + y^2$. Find the derivative of the composition $f\left(\vec{x}(t)\right)$.

<u>Solution I</u>

$$\frac{d}{dt} f(\vec{x}(t)) = \nabla f \circ \frac{d}{dt} \vec{x}(t) =$$

$$= \begin{bmatrix} 2 \cdot 2\cos t \\ 2\sin t \end{bmatrix} \circ \begin{bmatrix} -2\sin t \\ \cos t \end{bmatrix} = -8\cos t \sin t + 2\sin t \cos t =$$

$$= -3\sin 2t.$$

<u>solution II</u>

$$f(\vec{x}(t)) = f(2\cos t, \sin t) = 4\cos^2 t + \sin^2 t = 3\cos^2 t + 1$$

$$\frac{d}{dt}\left(3\cos^2 t + 1\right) = 3 \cdot 2\cos t(-\sin t) = -3\sin 2t.$$

Problem 9. (10 pts.) Let P be the xz- plane in $\mathbb{R}^3$.

a) Let T_1 be the linear transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ which reflects every vector across the plane P. What are the eigenvectors and eigenvalues of T_1?

Basis of xz - plane: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ - those vectors are left the same.

i.e. they are e-vectors corresponding to $\lambda = 1$

Basis of line perpendicular to xz - plane: $\begin{bmatrix}0\\1\\0\end{bmatrix}$. This vector gets flipped by the plane, i.e. it is e-vector corresponding to e-value $\lambda = -1$.

b) Let T_2 be the linear transformation $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ which orthogonally projects every vector to the plane P. What are the eigenvectors and eigenvalues of T_2?

As before:

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ correspond to $\lambda = 1$

but $\begin{bmatrix}0\\1\\0\end{bmatrix}$ gets send to zero, so it corresponds to $\lambda = 0$.

Problem 10. (10 pts.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{(x^3y + y^3x)}{(x^4 + y^4)} & \text{for } (x,y) \neq (0,0) \\ a & \text{for } (x,y) = (0,0) \end{cases}$$

What, if any, value of a will make $f(x,y)$ a continuous function?

a should be equal to $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y + y^3x}{x^4 + y^4}$

but on the line $y = kx$

$$f(x, kx) = \frac{x^3 \cdot kx + k^3x^4}{x^4 + k^4x^4} = k \cdot \frac{1+k^2}{1+k^4}$$

which has different values for different k's, thus the limit doesn't exists and there is no such $\boxed{a}$