**Problem 1.** (10 pts.) Mark as TRUE/FALSE the following statements. If a statement is false, give a simple example. If a statement is true, give a justification.

a) If $W$ and $V$ are linear subspaces of $\mathbb{R}^n$, then $\dim V + \dim W \le n$. TRUE FALSE

could be $W = V = \mathbb{R}^n$

and then $\dim W + \dim V = 2n > n$

b) If $A$ is a $k \times n$ matrix, then $\dim N(A) \le k$. TRUE FALSE

$\dim N(A) = \#$ of columns w/o pivot

so take

$$A = [\underbrace{0, 0, \dots}_{\text{as many as you wish}}, 0]$$

here $k = 1$

but $\dim N(A) = n$, which can be arbitrary large.

c) If $\vec{v} \neq \vec{0}$ and $V = \text{Span}\,(\vec{v})$, then $\dim V = 1$. TRUE FALSE

$\vec{v}$ is the only basis vector

d) Let $A$ be a $2 \times 3$ matrix. Then $\dim N(A) \geq 1$. TRUE FALSE

dim N(A) = # of columns w/o pivot and there are 3 columns with at most 2 pivots (in one row there may be at most one pivot).

**Problem 2.** Show that if $\{\vec{v}, \vec{w}\}$ is a basis of a subspace $S$, then $\{\vec{v}+\vec{w}, \vec{v}-\vec{w}\}$ is also a basis of $S$.

· $S = \text{Span}\{\vec{v}, \vec{w}\}$

We need to show: 1. $\vec{v}+\vec{w}$, $\vec{v}-\vec{w}$ are linearly independent.

2. $S = \text{Span}(\vec{v}+\vec{w}, \vec{v}-\vec{w})$.

Ad. 1

$$c_1(\vec{v}+\vec{w}) + c_2(\vec{v}-\vec{w}) =$$

$$(c_1+c_2)\vec{v} + (c_1-c_2)\vec{w} = \vec{0}$$ if and only if

$c_1+c_2=0$ and $c_1-c_2=0$ (as $\vec{v}, \vec{w}$ are independent).

$$\Downarrow$$

$$c_1 = c_2 = 0$$

$$\Downarrow$$

$\vec{v}+\vec{w}$ and $\vec{v}-\vec{w}$ are independent.

Ad 2. We'll use the fact that $\text{Span}(\vec{v}_1, \vec{v}_2) = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{u}$ for $\vec{u}$ = any linear combination of $\vec{v}_1, \vec{v}_2$.

$\text{Span}(\vec{v}+\vec{w}, \vec{v}-\vec{w}) = \text{Span}(\vec{v}+\vec{w}, \vec{v}-\vec{w}, \vec{v}) =$ | as $\vec{v} = \frac{1}{2}((\vec{v}+\vec{w}) + \frac{1}{2}(\vec{v}-\vec{w})$

$= \text{Span}(\vec{v}+\vec{w}, \vec{v})$ as $\vec{v}-\vec{w} = 2\vec{v} - (\vec{v}+\vec{w})$

$= \text{Span}(\vec{v}+\vec{w}, \vec{v}, \vec{w})$ as $\vec{w} = (\vec{v}+\vec{w}) - \vec{v}$

$= \text{Span}(\vec{v}, \vec{w})$ as $\vec{v}+\vec{w} = \vec{v}+\vec{w}$.

$= S$.

**Problem 3.** (10 pts.) Let $\vec{u} = [1, -1, 1, -1]^T$ and $\vec{w} = [0, 3, 3, 1]^T$.
a) Find the cosine of the angle between the vectors $\vec{u}$ and $\vec{w}$.

$$\cos \alpha = \frac{\vec{u} \circ \vec{w}}{|\vec{u}| \cdot |\vec{w}|}$$

$$\vec{u} \circ \vec{w} = -1$$

$$|\vec{u}|^2 = 4 \qquad |\vec{w}|^2 = 9 + 9 + 1 = 19$$

$$\boxed{\cos \alpha = \frac{-1}{2 \cdot \sqrt{19}}}$$

b) Find the numbers $a$ and $b$ such that the vector $[2, 4, a, b]^T$ is in the Span $(\vec{u}, \vec{w})$.

$c_1 \vec{u} + c_2 \vec{w} = [2, 4, a, b]$ should have a solution.

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 3 & 4 \\ 1 & 3 & a \\ -1 & 1 & b \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 3 & 6 \\ 0 & 3 & a-2 \\ 0 & 1 & b+2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & a-2-6 \\ 0 & 0 & b+2-2 \end{array}\right]$$

<u>So</u>: $\begin{cases} a = 8 \\ b = 0 \end{cases}$

**Problem 4.** (10 pts.) Let $A$ and $B$ be some matrices.
a) The spaces $N(A)$ and $N(B)$ are linear subspaces of the same $\mathbb{R}^n$ if (choose one that applies):

- The number of rows of $A$ equals to the number of rows of $B$.
- The number of columns of $A$ equals to the number of rows of $B$.
- The number of columns of $A$ equals to the number of columns of $B$.
- The number of rows of $A$ equals to the number of columns of $B$.

b) Assuming that $N(A)$ and $N(B)$ are linear subspaces of the same $\mathbb{R}^n$, determine if the space $V = N(A) \cap N(B)$ is a linear subspace of $\mathbb{R}^n$. If it is, show that three subspace properties are satisfied. If it is not, show by example that one of the properties fails.

Note: You may use the fact that the space $V = \{\vec{x} \,|A\,\vec{x} = \vec{0}$ and $B\,\vec{x} = \vec{0}\}$.

1) $\vec{0} \in V$ as $A \cdot \vec{0} = \vec{0}$ and $B \cdot \vec{0} = \vec{0}$.

2) $\vec{x} \in V \Rightarrow c \cdot \vec{x} \in V$

if $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$, then

$c \cdot A\vec{x} = A \cdot c\vec{x} = \vec{0}$ and $c \cdot B\vec{x} = B \cdot c\vec{x} = \vec{0}$

So: $c \cdot \vec{x} \in V$

3) $\vec{x}, \vec{y} \in V \Rightarrow \vec{x} + \vec{y} \in V$

If: $A\vec{x} = \vec{0}$, $A\vec{y} = \vec{0}$, $B\vec{x} = \vec{0}$ and $B\vec{y} = \vec{0}$, then:

$\vec{0} + \vec{0} = A\vec{x} + A\vec{y} = A(\vec{x} + \vec{y})$
$\vec{0} + \vec{0} = B\vec{x} + B\vec{y} = B(\vec{x} + \vec{y})$
$\Rightarrow A(\vec{x} + \vec{y}) = \vec{0}$ and $B(\vec{x} + \vec{y}) = \vec{0}$

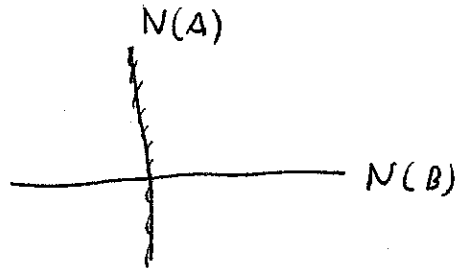
$\Rightarrow \vec{x} + \vec{y} \in V$

c) Let's take $A = [1, 0]$ and $B = [0, 1]$. Determine if the union $N(A) \cup N(B)$ is a linear subspace of $\mathbb{R}^2$.

**Note:** You may use the fact that the union is the space $\{\vec{x}\,|A\,\vec{x} = \vec{0} \text{ or } B\,\vec{x} = \vec{0}\}$.

$$N(A) = \{\vec{x} \mid x_1 = 0\}$$

$$N(B) = \{\vec{x} \mid x_2 = 0\}$$




then: $\vec{e}_2 \in N(A) \subset N(A) \cup N(B)$

$\vec{e}_1 \in N(B) \subset N(A) \cup N(B)$

but $A(\vec{e}_1 + \vec{e}_2) = 1$

and $B(\vec{e}_1 + \vec{e}_2) = 1$

so: $\vec{e}_1 + \vec{e}_2 \notin N(A) \cup N(B)$

so $N(A) \cup N(B)$ is <u>not</u> a linear subspace.

**Problem 5.** Let $\vec{v} = [1, 0, -1]^T$.

a) Find a basis for a linear subspace $V = \{\vec{x} \in \mathbb{R}^3 | \vec{x} \cdot \vec{v} = 0\}$.

$V = N(\tilde{A})$ for $\tilde{A} = [1, 0, -1]$

this matrix has one pivot, so $x_2$, $x_3$ are free variables and

$$N(\tilde{A}) = \text{Span}\left( \underset{x_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{x_3}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \right)$$

b) Find a matrix $A$ such that $N(A) =$ Span $(\vec{v})$.

The two linearly independent vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ together with $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ form a basis of $\mathbb{R}^3$.

So take $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ which is formed from transposing the basis of $\tilde{A}$.

$$A \cdot \left( c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_3 \cdot 0 = 0$$

if and only if $c_1 = c_2 = 0$

i.e. $N(A) = \{ c_3 \cdot \vec{v} \mid c_3 \in \mathbb{R} \} = \text{Span}(\vec{v})$.

c) Find a matrix $A$ such that $C(A) =$ Span $(\vec{v})$.

$$A = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

**Problem 6.** For a given matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & -2 & -3 \\ 3 & -6 & -2 & -4 & -5 \\ 3 & -6 & -4 & -8 & -13 \\ 2 & -4 & -1 & -1 & -2 \end{bmatrix} \quad \text{with} \quad \text{rref}\,(A) = \begin{array}{c} \begin{array}{ccccc} \phantom{1} & x_2 & \phantom{0} & \phantom{0} & x_5 \end{array} \\ \begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(you don't have to verify that $rref(A)$ is equal to the above matrix.)
a) find a basis of $N(A)$.

free: $x_2$, $x_5$

basis of $N(A)$: $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}$

b) Given that $A\,[1,1,0,0,0]^T = [-1,-3,-3,-2]^T$ find **all** solutions to

$$A\,\vec{x} = [-1,-3,-3,-2]^T$$

$$\vec{X} = \vec{X}_p + \vec{X}_h = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

c) find a basis of $C(A)$. = columns of A corresponding to columns with pivot in rref (A)

$$\begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -8 \\ -1 \end{bmatrix}$$

**Problem 7.** (10 pts.) Give examples of matrices with the following properties, or give a short explanation of why it is impossible:
(a) A $3 \times 6$ matrix $A$ with rank(A)=nullity(A)=3.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

(b) A $6 \times 3$ matrix $A$ with rank(A)=nullity(A)=3.

$\text{rk} A = 3 \Rightarrow$ there are 3 columns with pivot in rref (A)

$n(A) = 3 \Rightarrow$ there are 3 columns without pivot in rref(A)

$\Rightarrow$ # of columns of A must be 6, <u>not 3</u>

(c) A $1 \times 2$ matrix $A$ with $N(A) = \{\vec{0}\}$.

$N(A) = \{\vec{0}\} \Rightarrow$ there are no columns of rref $A$ without pivot, but each row may contain at most one pivot so:

$n(A) = 2 - \#\text{ columns with pivot} \geq 2 - 1 = 1$

So it can't be 0.

(d) A $2 \times 1$ matrix $A$ with $N(A) = \{\vec{0}\}$.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

**Problem 8.** (10 pts.) Let $\vec{w}$ and $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{97}}$ be some vectors in $\mathbb{R}^n$ such that $\vec{w}$ **does not** belong to Span $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{97}})$.

a) Is it possible that $n = 70$? Explain your answer. Does the answer change if the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{97}}$ are linearly independent?

Yes, it could be that $\vec{v}_1 = \vec{v}_2 = \cdots = \vec{v}_{97}$ and then it's easy for $\vec{w}$ not to belong to Span$(\vec{v}_1)$ in $\mathbb{R}^{70}$.

On the other hand if $\vec{v}_i$'s are independent, there may be at most 70 linearly independent vectors in $\mathbb{R}^{70}$, so it's <u>not</u> possible.

b) Let

$$A = \begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_{97}} \\ | & | & & | \end{bmatrix}$$

Under the same assumption that $\vec{w}$ **does not** belong to Span $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{97}})$, write all solutions to the equation:

$$A\vec{x} = \vec{w}$$

Since $\vec{w} \notin \text{Span}(\vec{v}_1, \ldots \vec{v}_{97}) = C(A)$ then the above equation has <u>no</u> <u>solutions</u>.

**Problem 9.** (10 pts.) Let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ be some vectors, and let

$$\begin{aligned} \vec{w_1} &= \vec{v_1} + 4\vec{v_2} + \vec{v_3} \\ \vec{w_2} &= 2\vec{v_1} - \vec{v_2} + \vec{v_3} \\ \vec{w_3} &= 5\vec{v_1} + \vec{v_2} - 2\vec{v_3} \end{aligned}$$

Let

$$A = \begin{bmatrix} | & | & | & | & | \\ \vec{w_1} & \vec{v_1} & \vec{v_2} & \vec{w_2} & \vec{v_3} \\ | & | & | & | & | \end{bmatrix}.$$

Write a particular solution (i.e. just one solution) to the equation:

$$A\vec{x} = \vec{w_3}$$

$$\begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

**Problem 10.** (10 pts.) Let $\vec{u} = [1, 1, 2, 3]^T$ and $\vec{v} = [1, -1, 7, 1]^T$. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be given by the formula:

$$T(\vec{x}) = \begin{bmatrix} \vec{u} \cdot \vec{x} \\ \vec{v} \cdot \vec{x} \end{bmatrix}$$

a) Show that $T$ is linear.

$$1)\quad T(c\vec{x}) = \begin{bmatrix} \vec{u} \cdot (c\vec{x}) \\ \vec{v} \cdot (c\vec{x}) \end{bmatrix} = \begin{bmatrix} c \cdot \vec{u} \cdot \vec{x} \\ c \cdot \vec{v} \cdot \vec{x} \end{bmatrix} = c \cdot \begin{bmatrix} \vec{u} \cdot \vec{x} \\ \vec{v} \cdot \vec{x} \end{bmatrix} = c \cdot T\vec{x}$$

$$2)\quad T(\vec{x} + \vec{y}) = \begin{bmatrix} \vec{u} \cdot (\vec{x} + \vec{y}) \\ \vec{v} \cdot (\vec{x} + \vec{y}) \end{bmatrix} = \begin{bmatrix} \vec{u}\vec{x} \\ \vec{v}\vec{x} \end{bmatrix} + \begin{bmatrix} \vec{u}\vec{y} \\ \vec{v}\vec{y} \end{bmatrix} = T\vec{x} + T\vec{y}$$

b) Find the matrix of $T$.

$$A = [T\vec{e}_1, T\vec{e}_2, T\vec{e}_3, T\vec{e}_4] = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & -1 & 7 & 1 \end{bmatrix}$$