Math 51 - Autumn 2007 - Midterm Exam I

Name: ______________________________

Student ID: ______________________________

Select your section:

Kirsten Wickelgren 03 (11:00-11:50 AM) 20 (10:00-10:50AM)	Jon Lee 08 (11:00-11:50AM) 26 (10:00-10:50AM)	Jesse Gell-Redman 11 (1:15-2:05PM) 24 (2:15-3:05PM)	Jason Miller 23 (1:15-2:05PM) 30 (2:15-3:05PM)	Josh Genauer 12 (1:15-2:05PM) 15 (11:00-11:50AM)
Dimitry Ivanov 14 (11:00-11:50AM) 29 (1:15-2:05PM)	Olena Bormashenko 17 (1:15-2:05PM) 21 (11:00-11:50AM)	Penka Georgieva 18 (1:15-2:05PM) 27 (11:00-11:50AM)	Robin Koycheff 02 (11:00-11:50 AM) 05 (1:15-2:05PM)	Jack Hall 09 (11:00-11:50AM) 06 (1:15-2:05PM)

Signature: ______________________________

Instructions: Print your name and student ID number, print your section number and TA's name, write your signature to indicate that you accept the honor code. During the test, you may not use notes, books, calculators. Read each question carefully, and show all your work.
There are ten problems on the pages numbered from 1 to 14, with the total of 100 points. Point values are given in parentheses. You have 2 hours (until 9PM) to answer all the questions.

In the exam all vectors are columns, but sometimes we use transpose to write them horizontally.

Thus $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = [v_1, v_2, \ldots, v_k]^T$.

Similarly $\vec{v}^T$ is a row $[v_1, v_2, \ldots, v_k]$.

The dot product of two vectors is denoted as $\vec{v} \cdot \vec{w}$.

Problem 1. (10 pts.) Mark as TRUE/FALSE the following statements. If a statement is false, give a simple example. If a statement is true, give a justification.

a) If W and V are linear subspaces of $\mathbb{R}^n$, then $\dim V + \dim W \leq n$. TRUE FALSE

b) If A is a $k \times n$ matrix, then $\dim N(A) \leq k$. TRUE FALSE

c) If $\vec{v} \neq \vec{0}$ and $V = \text{Span}\ (\vec{v})$, then $\dim V = 1$. TRUE FALSE

d) Let A be a 2×3 matrix. Then $\dim N(A) \geq 1$. TRUE FALSE

Problem 2. (10 pts.) Show that if $\{\vec{v}, \vec{w}\}$ is a basis of a subspace S, then $\{\vec{v} + \vec{w}, \vec{v} - \vec{w}\}$ is also a basis of S.

Problem 3. (10 pts.) Let $\vec{u} = [1, -1, 1, -1]^T$ and $\vec{w} = [0, 3, 3, 1]^T$.
a) Find the cosine of the angle between the vectors $\vec{u}$ and $\vec{w}$.

b) Find the numbers a and b such that the vector $[2, 4, a, b]^T$ is in the Span $(\vec{u}, \vec{w})$.

Problem 4. (10 pts.) Let A and B be some matrices.
a) The spaces $N(A)$ and $N(B)$ are linear subspaces of the same $\mathbb{R}^n$ if (choose one that applies):

- The number of rows of A equals to the number of rows of B.
- The number of columns of A equals to the number of rows of B.
- The number of columns of A equals to the number of columns of B.
- The number of rows of A equals to the number of columns of B.

b) Assuming that $N(A)$ and $N(B)$ are linear subspaces of the same $\mathbb{R}^n$, determine if the space $V = N(A) \cap N(B)$ is a linear subspace of $\mathbb{R}^n$. If it is, show that three subspace properties are satisfied. If it is not, show by example that one of the properties fails.

Note: You may use the fact that the space $V = \{\vec{x} \,|A\vec{x} = \vec{0} \text{ and } B\vec{x} = \vec{0}\}$.

c) Let's take $A = [1, 0]$ and $B = [0, 1]$. Determine if the union $N(A) \cup N(B)$ is a linear subspace of $\mathbb{R}^2$.

Note: You may use the fact that the union is the space $\{\vec{x}\,|A\,\vec{x} = \vec{0}$ or $B\,\vec{x} = \vec{0}\}$.

Problem 5. (10 pts.) Let $\vec{v} = [1, 0, -1]^T$.
a) Find a basis for a linear subspace $V = \{\vec{x} \in \mathbb{R}^3 | \, \vec{x} \cdot \vec{v} = 0\}$.

b) Find a matrix A such that $N(A) =$ Span $(\vec{v})$.

c) Find a matrix A such that $C(A) = \text{ Span } (\vec{v})$.

Problem 6. (10 pts.) For a given matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & -2 & -3 \\ 3 & -6 & -2 & -4 & -5 \\ 3 & -6 & -4 & -8 & -13 \\ 2 & -4 & -1 & -1 & -2 \end{bmatrix} \qquad \text{with} \qquad \text{rref}\,(A) = \begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(you don't have to verify that $rref(A)$ is equal to the above matrix.)
a) find a basis of $N(A)$.

b) Given that $A\,[1,1,0,0,0]^T = [-1,-3,-3,-2]^T$ find **all** solutions to

$$A\,\vec{x} = [-1,-3,-3,-2]^T$$

c) find a basis of $C(A)$.

Problem 7. (10 pts.) Give examples of matrices with the following properties, or give a short explanation of why it is impossible:
(a) A 3×6 matrix A with rank(A)=nullity(A)=3.

(b) A 6×3 matrix A with rank(A)=nullity(A)=3.

(c) A 1×2 matrix A with $N(A) = \{\vec{0}\}$.

(d) A 2×1 matrix A with $N(A) = \{\vec{0}\}$.

Problem 8. (10 pts.) Let $\vec{w}$ and $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{97}}$ be some vectors in $\mathbb{R}^n$ such that $\vec{w}$ **does not** belong to Span $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{97}})$.

a) Is it possible that $n = 70$? Explain your answer. Does the answer change if the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{97}}$ are linearly independent?

b) Let

$$A = \begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_{97}} \\ | & | & & | \end{bmatrix}$$

Under the same assumption that $\vec{w}$ **does not** belong to Span $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{97}})$, write all solutions to the equation:

$$A\,\vec{x} = \vec{w}$$

Problem 9. (10 pts.) Let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ be some vectors, and let

$$\begin{aligned} \vec{w_1} &= \vec{v_1} + 4\vec{v_2} + \vec{v_3} \\ \vec{w_2} &= 2\vec{v_1} - \vec{v_2} + \vec{v_3} \\ \vec{w_3} &= 5\vec{v_1} + \vec{v_2} - 2\vec{v_3} \end{aligned}$$

Let

$$A = \begin{bmatrix} | & | & | & | & | \\ \vec{w_1} & \vec{v_1} & \vec{v_2} & \vec{w_2} & \vec{v_3} \\ | & | & | & | & | \end{bmatrix}$$

Write a particular solution (i.e. just one solution) to the equation:

$$A\vec{x} = \vec{w_3}$$

Problem 10. (10 pts.) Let $\vec{u} = [1, 1, 2, 3]^T$ and $\vec{v} = [1, -1, 7, 1]^T$. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be given by the formula:

$$T(\vec{x}) = \begin{bmatrix} \vec{u} \cdot \vec{x} \\ \vec{v} \cdot \vec{x} \end{bmatrix}$$

a) Show that T is linear.

b) Find the matrix of T.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100