MATH 51 MIDTERM 2 SOLUTIONS

March 9, 2006

1(a). Find the determinant of the matrix $C = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Solution: for example,

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & -3 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 7 \end{vmatrix} = 7$$

1(b). Find the inverse of the matrix $M = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 1 & 8 \end{bmatrix}$. Ans:

$$\begin{bmatrix} 1 & 0 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 0 & 1 & 0 \\ -2 & 1 & 8 & | & 0 & 0 & 1 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 0 & 1 & 0 \\ 0 & 1 & 4 & | & 2 & 0 & 1 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 2 & -1 & 1 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & | & 5 & -2 & 2 \\ 0 & 1 & 0 & | & -6 & 4 & -3 \\ 0 & 0 & 1 & | & 2 & -1 & 1 \end{bmatrix}$$

so

$$M^{-1} = \begin{bmatrix} 5 & -2 & 2 \\ -6 & 4 & -3 \\ 2 & -2 & 1 \end{bmatrix}.$$

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation such that

(i) $T(\mathbf{v}_1) = 3\mathbf{v}_1 + 2\mathbf{v}_2$

(ii) $T(\mathbf{v}_2) = -\mathbf{v}_1 - \mathbf{v}_2$.

2(a). What is the matrix for T with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$?

Solution: The first column of the matrix is $[T(\mathbf{v}_1)]_{\mathcal{B}}$, or $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. The second column is $[T(\mathbf{v}_2)]_{\mathcal{B}}$, or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus the matrix is $B = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$.

2(b). What is the matrix for T with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$?

Solution: The change of basis matrix is $C = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ (the matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$.) The matrix for T in standard coordinates is $A = CBC^{-1}$. Now $C^{-1} = \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix}$, so

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} -2/3 & -1/3 \\ 7/3 & 8/3 \end{bmatrix}$$

3(a). Find all eigenvalues of the matrix $A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 1 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

Solution:

$$\begin{aligned} 0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -4 & -3 \\ -1 & \lambda - 1 & -2 \\ 0 & 0 & \lambda - 7 \end{vmatrix} &= (\lambda - 7) \begin{vmatrix} \lambda - 1 & -4 \\ -1 & \lambda - 1 \end{vmatrix} \\ = (\lambda - 7)\left((\lambda - 1)(\lambda - 1) - 4\right) &= (\lambda - 7)(\lambda^2 - 2\lambda - 3) \\ &= (\lambda - 7)(\lambda - 3)(\lambda + 1) \end{aligned}$$

so the eigenvalues are 7, 3, and -1.

3(b). The matrix $B = \begin{bmatrix} -1 & 8 \\ 8 & 11 \end{bmatrix}$ has characteristic polynomial

$$p(\lambda) = (\lambda - 15)(\lambda + 5).$$

Find a basis for $\mathbf{R}^2$ consisting of eigenvectors of B.

Solution: For $\lambda = 15$, we find a nonzero vector in the nullspace of $15I - B = \begin{bmatrix} 16 & -8 \\ -8 & 4 \end{bmatrix}$. Note $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is such a vector. (Any nonzero multiple would also do.)

We can get the second basis eigenvector in either of two ways:

(1) By symmetry of B, we know any eigenvector with eigenvalue -5 must be orthogonal to $\mathbf{v}_1$. Thus we can just rotate $\mathbf{v}_1$ by 90° to get our second eigenvector: $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(2) For $\lambda = -5$, we find a nonzero vector in the nullspace of $-5I - B = \begin{bmatrix} -4 & -8 \\ -8 & -16 \end{bmatrix}$. Note $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is such a vector.

3(c). Suppose C is a **symmetric** 2 x 2 matrix with determinant 5. Suppose the vector $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector with eigenvalue 3. Find an eigenvector $\mathbf{w}$ that is **not** a scalar multiple of $\mathbf{v}$, and find its eigenvalue. Explain.

Solution: By symmetry of C, we know C must have an orthonormal basis of eigenvectors. We can let $\mathbf{v}$ be one of the basis vectors, and we can rotate $\mathbf{v}$ by 90° to get a second eigenbasis vector: $\mathbf{w} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

The product of the eigenvalues is the determinant, so the eigenvalue of $\mathbf{w}$ must be $(\det C)/3 = 5/3$.

4. Let V be the subspace of $\mathbf{R}^3$ spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Consider the coordinate system for V determined by the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$.

4(a). Find the vector $\mathbf{w} \in \mathbf{R}^3$ whose expression in the $\mathcal{B}$-coordinate system is

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} 2 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (-1) = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

4(b). Find $[\mathbf{v}]_{\mathcal{B}}$ (the expression for $\mathbf{v}$ in the $\mathcal{B}$-coordinate system) for the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}.$$

Solution: Let $\mathbf{u} = [\mathbf{v}]_{\mathcal{B}}$. Then $\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$. Solving this system (by Gaussian elimination, for example) gives

$$\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

5. Consider the functions

$$u(x, y) = x^3 y^2$$
$$v(x, y) = y/x.$$

5(a). Find the following partial derivatives:

$$\frac{\partial u}{\partial x} = \boxed{3x^2y^2}$$
$$\frac{\partial u}{\partial y} = \boxed{2x^3y}$$
$$\frac{\partial v}{\partial x} = \boxed{-y/x^2}$$
$$\frac{\partial v}{\partial y} = \boxed{1/x}$$
$$\frac{\partial^2 u}{\partial x \partial y} = \boxed{6x^2y}$$

5(b). Suppose $f(u,v)$ is a function of u and v such that $\frac{\partial f}{\partial u} = 12 - v$ and $\frac{\partial f}{\partial v} = 1 - u$. Find

$$\frac{\partial}{\partial x} f(u(x,y), v(x,y))$$

at the point $(x,y) = (1,2)$.

Solution:

$$\begin{aligned}\frac{\partial}{\partial x} f(u(x,y), v(x,y)) &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\ &= (12 - v)\frac{\partial u}{\partial x} + (1-u)\frac{\partial v}{\partial x} \\ &= (12 - y/x)(3x^2y^2) + (1 - x^2y^2)(-y/x^2) \\ &= 36x^2y^2 - 3xy^3 - y/x^2 + y^3.\end{aligned}$$

Thus at the point $(x,y) = (1,2)$:

$$\frac{\partial f}{\partial x} = 36(1)^2(2)^2 - 3(1)(2)^3 - 2/(1)^2 + 2^3 = 144 - 24 - 2 + 8 = \boxed{126}$$

6. Find the following:

6(a). The matrix for $T : \mathbf{R}^3 \to \mathbf{R}^3$, where T is rotation by 180° about the y-axis followed by 180° rotation about the z-axis.

Solution: Let R_y and R_z denote the 180° rotations about the y and z axes, respectively. Then

$$\begin{aligned}T(\mathbf{e}_1) &= R_z(R_y\mathbf{e}_1) = R_z(-\mathbf{e}_1) = \mathbf{e}_1 \\ T(\mathbf{e}_2) &= R_z(R_y\mathbf{e}_2) = R_z(\mathbf{e}_2) = -\mathbf{e}_2 \\ T(\mathbf{e}_3) &= R_z(R_y\mathbf{e}_3) = R_z(-\mathbf{e}_3) = -\mathbf{e}_3\end{aligned}$$

so the matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

6(b). The matrix for the linear map T given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + 7y \\ 13y \\ x - 4y \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 3 & 7 \\ 0 & 13 \\ 1 & -4 \end{bmatrix}.$$

6(c). The matrix for reflection in $\mathbf{R}^2$ about the line $y = -x$.

Solution: $\mathbf{e}_1 \mapsto -\mathbf{e}_2$ and $\mathbf{e}_2 \mapsto -\mathbf{e}_1$, so the matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

6(d). The product $\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 7 \\ 2 & 1 & 1 \end{bmatrix}$.

Solution: $\begin{bmatrix} 3 & 1 & 8 \\ 3 & 0 & 21 \end{bmatrix}$.

7. Find the equation of the tangent plane to the surface

$$xy + yz + zx = 11$$

at the point $(1, 2, 3)$.

Solution: The surface is a level set of the function $f(x, y, z) = xy + yz + xz$.

$$\nabla f(x, y, z) = (f_x, f_y, f_z) = (y + z, x + z, x + y)$$

so

$$\mathbf{N} = \nabla f(1, 2, 3) = (2 + 3, 1 + 3, 1 + 2) = (5, 4, 3)$$

is a normal vector to the surface (and therefore also to the tangent plane) at $(1, 2, 3)$. Since the tangent plane passes through the point $(1, 2, 3)$, the equation is

$$\boxed{(5, 4, 3) \cdot ((x, y, z) - (1, 2, 3)) = 0}$$

or (equivalently)

$$\boxed{5(x - 1) + 4(y - 2) + 3(z - 3) = 0}$$

or

$$\boxed{5x + 4y + 3z = 22.}$$

8. Consider the function $F : \mathbf{R}^2 \to \mathbf{R}^2$ given by:

$$F(x,y) = \begin{bmatrix} xe^y + y + 3\cos y \\ 1 + 2x + e^x \sin y \end{bmatrix}.$$

8(a). Find the matrix derivative $DF(x,y)$.

Solution: $\dfrac{\partial F}{\partial x} = \begin{bmatrix} e^y \\ 2 + e^x \sin y \end{bmatrix}$ and $\dfrac{\partial F}{\partial y} = \begin{bmatrix} xe^y + 1 - 3\sin y \\ e^x \cos y \end{bmatrix}$, so

$$DF(x,y) = \begin{bmatrix} e^y & xe^y + 1 - 3\sin y \\ 2 + e^x \sin y & e^x \cos y \end{bmatrix}.$$

Note:

$$DF(0,0) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

8(b). Using derivatives, estimate $F(.002, .003)$.

Solution:

$$\begin{aligned} F(.003, .002) &\approx F(0,0) + DF(0,0) \begin{bmatrix} .003 \\ .002 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} .002 \\ .003 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} .005 \\ .007 \end{bmatrix} \\ &= \begin{bmatrix} 3.005 \\ 1.007 \end{bmatrix} \end{aligned}$$

8(c). Find a point (x,y) so that $F(x,y) \approx \begin{bmatrix} 3.004 \\ 1.007 \end{bmatrix}$.

Solution: We want to solve $F(x,y) = \begin{bmatrix} 3.004 \\ 1.007 \end{bmatrix}$. Approximating $F(x,y)$ using the derivative at $(0,0)$ as in part (b) gives the equation:

$$F(0,0) + DF(0,0) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.004 \\ 1.007 \end{bmatrix}$$

or

$$\begin{aligned} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 3.004 \\ 1.007 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} .004 \\ .007 \end{bmatrix} \end{aligned}$$

Solving by Gaussian elimination gives $x = .003$ and $y = .001$.