Math 51 Midterm I Solutions

(These 'solutions' are written more in the form of a guide to correct thinking, and a guide to the level of understanding we were trying to test. This is for students who want to learn from their mistakes and improve their understanding.)

PROBLEM 1(a). This is a routine row reduced echelon form problem. It tests a computational skill only. Here is the answer.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If you missed it, just check your arithmetic and see where you slipped up.

PROBLEM 1(b). The point here is that even though we don't know the entries a and b yet, we can start the rref process and get a lot of information. Remember, the final rref form will tell us the dimension of the null space, which will be the number of free variables.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & 3 & b \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & a-1 \\ 0 & 2 & b-1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & a-1 \\ 0 & 0 & 1+b-2a \end{pmatrix}$$

At this point you see that rref(A) will definitely have pivots in the first two rows and columns. One of two things happens after that. If $1+b-2a \neq 0$ then the rref will have three pivots in the end and $N(A) = \{\mathbf{0}\}$. If $1+b-2a = 0$ then the last row of rref(A) will be all 0's. There is then one free variable and the null space is a line in $\mathbf{R}^3$.

PROBLEM 2(i). There are two basic views of a matrix-vector product $A\mathbf{v}$. The first view is tested here. Multiply the respective columns of A by the entries of $\mathbf{v}$ and add. So here

$$A\mathbf{v} = 1\mathbf{a_1} + 2\mathbf{a_2} + 3\mathbf{a_3} + 4\mathbf{a_4}$$

(See the bottom of page 47 of the text.)

Notice this view makes it crystal clear that vectors of the form $A\mathbf{v}$ are exactly the vectors in the column space of A, that is, in the span of the columns of A

PROBLEM 2(ii). This is the second important view of a matrix-vector product. The entries of $A\mathbf{v}$ are obtained as dot products of the rows of A and the vector $\mathbf{v}$. Let's ignore the business with transposes and just treat a row as a vector, even though it is written sideways, not up and down. Then

$$A\mathbf{v} = \begin{pmatrix} \mathbf{r_1} \cdot \mathbf{v} \\ \mathbf{r_2} \cdot \mathbf{v} \\ \mathbf{r_3} \cdot \mathbf{v} \end{pmatrix}$$

.

(See the bottom of page 48 of the text.)

Notice this view makes it crystal clear that vectors $\mathbf{v}$ in the null space of A are exactly the vectors that are orthogonal to all rows of A.

PROBLEM 2(iii). The answer is 'no'. Any four vectors in $\mathbf{R}^3$ are linearly dependent because if you put those vectors in the columns of a 3 by 4 matrix, like A here, then rref(A) has at most three pivots, therefore there are free variables and hence non-zero vectors in $N(A)$. But any non-zero vector in $N(A)$ gives rise to a linear dependence relation between the columns of A, using the formula for $A\mathbf{v}$ as a linear combination of columns of A.

PROBLEM 2(iv). Infinitely many. The explanation just above in 2(iii) explains why the null space of A contains at least a line in $\mathbf{R}^4$. There is at least one free variable, maybe more.

PROBLEM 2(v). For a k by n matrix, the number of pivot columns in rref(A) plus the number of non-pivot columns is the total number of columns, namely n. The number of non-pivot columns (free variables) is the dimension of the null space of A, called p here. The number of pivot columns is the dimension of the column space of A, called q here. So $p + q = n$. This is the famous 'rank-nullity theorem', on page 77 of the text. In our case, $n = 4$. However, we also know $p = 1$, 2, 3 or 4, since there must be free variables. That is, p cannot be 0.

The answer is therefore (p, q) = (1, 3) or (2, 2) or (3, 1) or (4, 0).

(The last case happens only when every entry of the 3 by 4 matrix A is 0.)

PROBLEM 3(a). In one sense this is a routine computational problem. But there are a LOT of important ideas behind it. First of all,

$$\boxed{\textit{Null space of } A \;=\; \textit{Null space of } rref(A).}$$

Now, to find the null space of the matrix $rref(A)$ given here, we need to solve the system

$$\begin{aligned} 1x_1 + 2x_2 + 0x_3 + 3x_4 &= 0 \\ 0x_1 + 0x_2 + 1x_3 - 2x_4 &= 0. \end{aligned}$$

This system is very easy to solve: x_2 and x_4 can be any numbers. Then

$$\begin{aligned} x_1 &= -2x_2 - 3x_4 \\ x_3 &= 2x_4. \end{aligned}$$

Arranging this in a vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

This equation shows that the two vectors

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

span the null space of $rref(A)$, which is the same as the null space of A. But these two vectors are also linearly independent, as can be seen by focusing on the 2^{nd} and 4^{th} coordinates alone. Thus, these two vectors form a basis of the null space of A.

That solves the problem, but underlying theoretical ideas are very important, so we will take a closer look. WHY is $N(A) = N(rref(A))$? The point is, you start with a system of equations $A\mathbf{x} = \mathbf{0}$, then begin performing row operations to transform it to a simpler system with *exactly the same solutions* $\mathbf{x}$. The row operations never change the vector $\mathbf{0}$ on the right side of the equation. So, $A\mathbf{x} = \mathbf{0}$ has exactly the same solutions as $rref(A)\mathbf{x} = \mathbf{0}$. This is what it means for the two null spaces to be the same.

PROBLEM 3(b). I think only about 20 people got this right, out of 460. To begin with, what is a basis of the column space of A? It is any collection of *linearly independent* vectors that *spans* the column space. I'll give the solution to the specific question asked here in minute, but one of the most fundamental results of the whole subject is that the number of elements in a basis of a given subspace *cannot be varied*. If two vectors form a basis, then *every other basis* must also consist of exactly two vectors. This is discussed at great length in the section on bases and dimension of a subspace, pages 69-77 of the text. So, if you circled sets here with different numbers of elements in them, go back to GO, and read pages 69-77 at least eleven times, or until you get the concepts of basis and dimension. Although I'll admit the theorems about bases of subspaces in high dimensions are not easy, in this problem we are dealing with something very down to earth: bases of a certain plane in $\mathbf{R}^3$. You can see the picture. Any basis must have exactly two elements, you don't need to know fancy theorems to see that.

OK, that said, which sets of columns of A should we circle here? Many students memorize the statement 'A basis of the column space of A is given by the columns of A corresponding to the pivot columns of $rref(A)$'. But perhaps they have no idea why this is true, and maybe even no idea what a basis is. If we'd asked for 'a' basis, lots of people probably would have said $\{\mathbf{a_1}, \mathbf{a_3}\}$, because the first and third columns of $rref(A)$ are the pivot columns. We wanted to punish students who memorized the statement above. but didn't know what it said or meant, or how you find *other* bases. We succeeded.

To explain how to find *other* bases, you need to *understand* why the statement about pivot columns is true in the first place. Even though we are thinking about column spaces here, the key is that the null spaces of A and $rref(A)$ are the same, which I discussed after I gave the solution of Problem 3(a). A consequence of this fact about null spaces is that *any linear dependence relation between columns of $rref(A)$ is matched by exactly the same linear dependence relation between the corresponding columns of A*. Go back to Problem 2(i) to see this. The coordinates of a null space vector give multiples of the columns that add up to $\mathbf{0}$.

First consequence: If some collection of columns of $rref(A)$ are linearly independent, then the same collection of columns of A are linearly independent.

Second consequence: If some collection of columns of $rref(A)$ are linearly dependent, then the same collection of columns of A are linearly dependent.

Conclusion: For any set of columns of $rref(A)$ that give a basis of its column space, exactly the same columns of A will give a basis of its column space.

Now stare at $rref(A)$ in this problem. Any two columns, except the first and second, give a basis of the column space of $rref(A)$. Just look at them. Any such pair is linearly independent (a very easy check), and they also span the column space of $rref(A)$, which, after all, consists of all vectors of form

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in \mathbf{R}^3$$

.

So, there are five correct answers to circle. All pairs of columns of A except the first pair $\{\mathbf{a_1}, \mathbf{a_2}\}$. Those two satisfy the relation $\mathbf{a_2} = 2\mathbf{a_1}$, because that relation holds between the first and second columns of $rref(A)$.

PROBLEM 3(c). Mercifully, you don't need to read a long explanation here, I've already given it in the discussion of Problem 3(b). Since the fourth column of $rref(A)$ is 3 times the first minus 2 times the third, we also have $\mathbf{a_4} = 3\mathbf{a_1} - 2\mathbf{a_3}$. Think of it in terms of null spaces. $\begin{pmatrix} 3 \\ 0 \\ -2 \\ -1 \end{pmatrix}$ is in the null space of $rref(A)$, so it is also in the null space of A, which is exactly the statement $3\mathbf{a_1} - 2\mathbf{a_3} - \mathbf{a_4} = \mathbf{0}$.

PROBLEM 3(d). From Problem 2(i), you know $A \begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \end{pmatrix} = 2\mathbf{a_1} + 3\mathbf{a_4} = \mathbf{b}$.

Thus we have found one solution of $A\mathbf{x} = \mathbf{b}$. By the general theory, which everyone should know, *all solutions* of $A\mathbf{x} = \mathbf{b}$ are given by

$$\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

That is, we add one solution to all null space vectors from Problem 3(a).

PROBLEM 4(a). We must show that S contains $\mathbf{0}$ and is closed under sums and scalar multiples. Well, $\mathbf{0}$ is in W, and $\mathbf{0} \cdot \mathbf{v} = 0$ for all $\mathbf{v}$ in V. That takes care of the first thing, $\mathbf{0}$ is indeed in S. Next, suppose $\mathbf{t}$ and $\mathbf{u}$ both belong to S. Then they both belong to W and W is a subspace, so $\mathbf{t}+\mathbf{u}$ is in W. Also, for any $\mathbf{v}$ in V, $(\mathbf{t}+\mathbf{u})\cdot\mathbf{v} = \mathbf{t}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v} = 0+0 = 0$. This proves S is closed under sums. Finally, for any scalar c, $c\mathbf{u}$ is in W, because W is a subspace. Also, $c\mathbf{u}\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = c0 = 0$. So $c\mathbf{u}$ is in S. Now S has passed all three tests to be a subspace.

PROBLEM 4(b). Unlike Problem 3(b), we felt badly that more students didn't get this one. The difficulty was, although the wording describing the set H is completely unambiguous according to the way mathematicians use language, students haven't had as much experience reading such descriptions carefully. Spelling it out more, the set H consists of all the sums $\mathbf{v}+t\mathbf{x}$, where $\mathbf{v}$ varies over *all* of the infinitely many vectors on L and $t \geq 0$ varies over all non-negative scalars. For example, suppose L is the y-axis, consisting of all $\mathbf{v} = \begin{pmatrix} 0 \\ y \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $H =$ all $\begin{pmatrix} 0 \\ y \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} t \\ y \end{pmatrix}$, with any y and any $t \geq 0$. This is exactly the half plane, on and to the right of the y-axis. H passes the first two tests for a subspace, it contains $\mathbf{0}$ and it is closed under sums. But H fails the third test, negative multiples of vectors in H are not always in H.

If you think about the clarification of the wording above, no matter what is the line L and the vector $\mathbf{x}$ not on L, the set H will be a half plane, passing the first two tests for a subspace, but failing the third test about all scalar multiples.

PROBLEM 5(a). The columns of B will be the vectors $\mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{T}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Using the hint and the fact that $\mathbf{T}$ is a linear transformation,

$$\mathbf{T}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \mathbf{T}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So, the first column of B is

$$\mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}.$$

To find the second column of B, use $\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So

$$\mathbf{T}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{T}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus the second column of B is

$$\mathbf{T}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}. \text{ Summarizing, } B = \begin{pmatrix} -2 & 6 \\ -4 & 5 \end{pmatrix}.$$

PROBLEM 5(b). The theory is the same as the main point in part (a), namely, the columns of A will be the vectors $\mathbf{S}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{S}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These are, by definition of S, the vectors obtained by rotating the standard unit vectors θ radians counterclockwise. The answer, by elementary trig, is then

$$A = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}.$$

PROBLEM 5(c). The product of two matrices is defined so as exactly to be the matrix representing the composition of two linear transformations. So the matrix for $\mathbf{S} \circ \mathbf{T}$ will be the product

$$AB = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} -2 & 6 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 2 & -2/5 \\ -4 & 39/5 \end{pmatrix}.$$

PROBLEM 6(a). Any four vectors in $\mathbf{R}^3$ are linearly dependent. This was explained in the solution of 2(iii). Some words like those should have been included in your solution here. I'll repeat. Any four vectors in $\mathbf{R}^3$ are linearly dependent because if you put those vectors in the columns of a 3 by 4 matrix then the rref of the matrix has at most three pivots, therefore there are free variables and hence non-zero vectors in the null space of the matrix. But any non-zero vector in the null space gives rise to a linear dependence relation between the columns by using the formula of Problem 2(i).

PROBLEM 6(b). From part (a), we know there is a relation

$$c_1\mathbf{T}(\mathbf{v_1}) + c_2\mathbf{T}(\mathbf{v_2}) + c_3\mathbf{T}(\mathbf{v_3}) + c_4\mathbf{T}(\mathbf{v_4}) = \mathbf{0}$$

where not all c_j are 0. Now apply the linear transformation S, remembering that S takes sums to sums and scalar multiples to scalar multiples.

$$\mathbf{0} = S(\mathbf{0}) = S(c_1\mathbf{T}(\mathbf{v_1}) + c_2\mathbf{T}(\mathbf{v_2}) + c_3\mathbf{T}(\mathbf{v_3}) + c_4\mathbf{T}(\mathbf{v_4}))$$

$$= c_1\mathbf{S} \circ \mathbf{T}(\mathbf{v_1}) + c_2\mathbf{S} \circ \mathbf{T}(\mathbf{v_2}) + c_3\mathbf{S} \circ \mathbf{T}(\mathbf{v_3}) + c_4\mathbf{S} \circ \mathbf{T}(\mathbf{v_4}).$$

But this is exactly a linear dependence relation between the four vectors $\{\mathbf{S} \circ \mathbf{T}(\mathbf{v_1}), \mathbf{S} \circ \mathbf{T}(\mathbf{v_2}), \mathbf{S} \circ \mathbf{T}(\mathbf{v_3}), \mathbf{S} \circ \mathbf{T}(\mathbf{v_4})\}$.

PROBLEM 7(a). This is a routine computational problem. The needed formula here is

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}||\ cos(\theta)$$

where θ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$. This is the most important formula involving dot products, and should have been on the tip of your tongue, so to speak, as a tool for solving problems.

In this problem, $\mathbf{u}$ and $\mathbf{v}$, will be difference vectors from A to B and from A to C. Subtracting gives $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, -1, 2)$. Then

$$cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||} = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3}.$$

PROBLEM 7(b). We'll use the same main formula as in part (a), but combined with some arithmetic using other dot product properties.

$$||\mathbf{u}|| = \sqrt{(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})} = \sqrt{||\mathbf{x}||^2 + 2\mathbf{x} \cdot \mathbf{y} + ||\mathbf{y}||^2} = \sqrt{4 + 6 + 9} = \sqrt{19}$$

$$||\mathbf{v}|| = \sqrt{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})} = \sqrt{||\mathbf{x}||^2 - 2\mathbf{x} \cdot \mathbf{y} + ||\mathbf{y}||^2} = \sqrt{4 - 6 + 9} = \sqrt{7}$$

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} - \mathbf{y}\mathbf{y} = ||\mathbf{x}||^2 - ||\mathbf{y}||^2 = 4 - 9 = -5$$

Finally, if we call α the angle between $\mathbf{u}$ and $\mathbf{v}$, we get

$$cos(\alpha) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||} = \frac{-5}{\sqrt{7}\sqrt{19}}.$$