MATH 51 MIDTERM I October 19, 2006

Instructions:

- No calculators, books, notes, or electronic devices may be used during the exam.
- You have 90 minutes.
- There are 7 problems, each with multiple parts. You should work quickly so as to not leave out problems towards the end of the exam.
- Write solutions on the exam sheet. If extra space is needed use the back of a page.

Name:____
(print clearly)

Signature:____
(for acceptance of honor code)

Problem 1 (15 points)	
Problem 2 (15 points)	
Problem 3 (20 points)	
Problem 4 (15 points)	
Problem 5 (15 points)	
Problem 6 (10 points)	
Problem 7 (10 points)	
Total (100 points)	

Your TA/discussion section (circle one):

Antebi (15, 18)	Ayala (3, 6)	Easton (14, 17)
Fernanadez (2, 5)	Kim (8, 11)	Koytcheff (9, 12)
Lo (21, 24)	Rosales (26, 27)	Tzeng (20, 23)
Zamfir (29, 30)	Schultz (51A)	

1. (a) Find the reduced echelon form of $\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 & 3 \\ 1 & 0 & 1 & 2 & 3 \end{bmatrix}$.

(b) Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & 3 & b \end{bmatrix}$ where a and b are real numbers. Show that the null space of A is either $\{\mathbf{0}\}$ or a line, and give conditions on a and b that guarantee the null space of A is a line.

2. [short answer] Let A be a 3×4 matrix (3 rows, 4 columns) with columns $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4 \in \mathbf{R}^3$ and rows $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3 \in \mathbf{R}^4$, i.e. $A = \begin{bmatrix} | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} - & \mathbf{r}_1^T & - \\ - & \mathbf{r}_2^T & - \\ - & \mathbf{r}_3^T & - \end{bmatrix}$. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \mathbf{R}^4$.

(i) Express $A\mathbf{v}$ in terms of the columns of $\mathbf{a}_j$ of A.

(ii) Express $A\mathbf{v}$ in terms of the rows $\mathbf{r}_i$ of A.

(iii) Can the columns of A be linearly independent?

(iv) How many solutions $\mathbf{x}$ are there to the system $A\mathbf{x} = \mathbf{0}$? None, one, infinitely many, or does it depend on A?

(v) Find all possible pairs of numbers (p, q) so that p is the dimension of the null space of A and q is the dimension of the column space of A.

3. A certain 3×4 matrix A with every entry non-zero has reduced echelon form

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(Note that rref(A) and A are not the same matrix.)

(a) Find a basis for the null space of A.

(b) If the columns of A, in order, are $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4 \in \mathbf{R}^3$, circle *all* sets of vectors below that give a basis for the column space of A.

$\{\mathbf{a}_1, \mathbf{a}_2\}$ $\{\mathbf{a}_1, \mathbf{a}_3\}$ $\{\mathbf{a}_1, \mathbf{a}_4\}$ $\{\mathbf{a}_2, \mathbf{a}_3\}$ $\{\mathbf{a}_2, \mathbf{a}_4\}$ $\{\mathbf{a}_3, \mathbf{a}_4\}$

$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$

$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$

(3. continued)

(c) Find a linear dependence relation between the columns $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ of A, and explain your answer.

(d) If $\mathbf{b} = 2\mathbf{a}_1 + 3\mathbf{a}_4 \in \mathbf{R}^3$, where $\mathbf{a}_1$ and $\mathbf{a}_4$ are the first and fourth columns of A, find *all solutions* $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$. [Hint: What is *one* solution?]

4. (a) Assume that V and W are linear subspaces of $\mathbf{R}^n$. Recall that vectors $\mathbf{a}$, $\mathbf{b} \in \mathbf{R}^n$ are *orthogonal* if $\mathbf{a} \cdot \mathbf{b} = 0$. Let S be the set of all vectors $\mathbf{w} \in W$ that are orthogonal to every vector $\mathbf{v} \in V$. Prove that S is a linear subspace of $\mathbf{R}^n$.

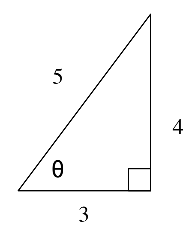
(b) Suppose $L \subset \mathbf{R}^2$ is a line through $\mathbf{0}$. Let $\mathbf{x} \in \mathbf{R}^2$ be a vector not in L. Let H be the set of all vectors of the form $\mathbf{v} + t\mathbf{x}$, where $\mathbf{v} \in L$ and $t \geq 0$ is a nonnegative scalar.

Draw a picture of H. Is H a linear subspace of $\mathbf{R}^2$? If yes, prove it. If no, which of the defining conditions for a subspace hold for H and which fail?

5. (a) Let $\mathbf{T} : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation with $\mathbf{T}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\mathbf{T}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Find a matrix B so that $B\mathbf{x} = \mathbf{T}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^2$. [Hint: What is $\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?]

(b) Let $\mathbf{S} : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that rotates vectors about the origin θ radians counterclockwise, where θ is the angle adjacent to the side of length 3 in a 3, 4, 5 right triangle. Find a matrix A so that $A\mathbf{x} = \mathbf{S}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^2$.



(c) With $\mathbf{T}$ and $\mathbf{S}$ as in parts (a) and (b), find a matrix C so that $C\mathbf{x} = (\mathbf{S} \circ \mathbf{T})(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^2$.

6. (a) Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be four vectors in $\mathbf{R}^5$ and let $\mathbf{T} : \mathbf{R}^5 \to \mathbf{R}^3$ be a linear transformation. Explain why the four vectors $\{\mathbf{T}(\mathbf{v}_1), \mathbf{T}(\mathbf{v}_2), \mathbf{T}(\mathbf{v}_3), \mathbf{T}(\mathbf{v}_4)\}$ are linearly dependent.

(b) Let $\mathbf{S} : \mathbf{R}^3 \to \mathbf{R}^5$ be another linear transformation. Explain why the four vectors $\{(\mathbf{S} \circ \mathbf{T})(\mathbf{v}_1), (\mathbf{S} \circ \mathbf{T})(\mathbf{v}_2), (\mathbf{S} \circ \mathbf{T})(\mathbf{v}_3), (\mathbf{S} \circ \mathbf{T})(\mathbf{v}_4)\}$ in $\mathbf{R}^5$ must be linearly dependent, where $\mathbf{T} : \mathbf{R}^5 \to \mathbf{R}^3$ is the linear transformation from part (a).

7. (a) Consider the triangle in $\mathbf{R}^3$ with vertices $A(1,2,3)$, $B(2,3,4)$, and $C(2,1,5)$. Find the cosine of the angle of the triangle at vertex A.

(b) Suppose $\mathbf{x}$, $\mathbf{y}$ are vectors in $\mathbf{R}^{10}$ and suppose $\mathbf{x} \cdot \mathbf{y} = 3$, $\|\mathbf{x}\| = 2$, and $\|\mathbf{y}\| = 3$. Find the cosine of the angle between $\mathbf{u} = \mathbf{x} + \mathbf{y}$ and $\mathbf{v} = \mathbf{x} - \mathbf{y}$.