

Math 51- Fall 2006 - Final Exam Solutions

1. Suppose $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and suppose you know that $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 9 \end{bmatrix}$.

(a) [5] Write in parametric form all solutions of the system of equations $Ax = \begin{bmatrix} -1 \\ 7 \\ 9 \end{bmatrix}$.

The solutions will be a translation of the nullspace of A by a particular solution $\vec{x}_p \in \mathbb{R}^5$.

We first note that $N(A) = N(\text{rref}(A)) = \left\{ \vec{x} \in \mathbb{R}^5 \mid \begin{array}{l} x_1 + 2x_2 + x_4 + 2x_5 = 0 \\ x_3 + x_4 + 2x_5 = 0 \end{array} \right\}$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mid \begin{array}{l} x_1 = -2x_2 - x_4 - 2x_5 \\ x_3 = -x_4 - 2x_5 \end{array} \right\} = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

With $\vec{x}_p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ -5 \end{bmatrix}$, we get $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \mid t, u, v \in \mathbb{R} \right\}$ for the solution set.

(b) [5] Denote the i -th column of A by \vec{a}_i . Suppose $\vec{a}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ and $\vec{a}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Find A .

[Hint: Make use of some linear dependence relations between the columns of A .]

The dependencies among the columns of $\text{rref}(A)$ are the same as those among the columns of A , so for instance we know that $2\vec{a}_1 = \vec{a}_2$, and $\vec{a}_1 + \vec{a}_3 = \vec{a}_4$, and $2\vec{a}_1 + 2\vec{a}_3 = \vec{a}_5$.

Thus, $\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{a}_3 = \vec{a}_4 - \vec{a}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -4 \end{bmatrix}$,

and $\vec{a}_5 = 2\vec{a}_1 + 2\vec{a}_3 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$. (Note also $\vec{a}_5 = 2\vec{a}_4 = 2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.)

Thus $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & -3 & -1 & -2 \\ 3 & 6 & -4 & -1 & -2 \end{bmatrix}$.

2. For each of the following subsets S of \mathbf{R}^3 determine if S is a subspace of \mathbf{R}^3 . If *not*, give a reason. If S is a subspace you don't need to prove that, but give a *basis* of S .

$$(a) [2] S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x - 2y + 3z = 2 \right\}$$

Not a subspace; $\vec{0} \notin S$ because $0 - 2 \cdot 0 + 3 \cdot 0 \neq 2$.

$$(b) [4] S = \left\{ \text{All } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ orthogonal to both } \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$$

This is a subspace (it's orthogonal complement of $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}\right\}$, or equivalently the null space of $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$);

a basis of S is $\left\{ \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix} \right\}$.

$$(c) [4] S = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \right\}$$

This is a subspace, but there is a dependency in this spanning set:

$$\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Thus, S has as a basis the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

(These two vectors are linear ind. by virtue of not being scalar multiples of each other.)

3. (a) [4] For which choice(s) of constant k is the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$ not invertible?

The determinant $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{vmatrix} = -\begin{vmatrix} 1 & k \\ 1 & k^2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -(k^2 - k) + (4 - 2)$

$$= -k^2 + k + 2,$$

which is zero if and only if the matrix is not invertible.

Solve: $0 = -k^2 + k + 2 = -(k^2 - k - 2) = -(k - 2)(k + 1)$, i.e. $\boxed{k = 2, -1}$.

- (b) [3] Let $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$. Find $\det(A)$ and A^{-1} .

$$\det A = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \cos^2 \theta - (-\sin^2 \theta) = \cos^2 \theta + \sin^2 \theta = \boxed{1}.$$

Three ways to find A^{-1} : Apply the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Rightarrow \boxed{A^{-1} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}}$.

Or, row-reduce the augmented matrix $\left[\begin{array}{cc|cc} \cos \theta & -\sin \theta & 1 & 0 \\ \sin \theta & \cos \theta & 0 & 1 \end{array} \right]$ and take 3rd & 4th cols (same result);

or, note that A is the matrix of the transformation on \mathbb{R}^2 that rotates counterclockwise by θ , so that A^{-1} should be the matrix for rotation by $-\theta$: $A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$.

- (c) [3] If B is an $n \times n$ matrix, find a formula for $\det(3B)$ in terms of $\det(B)$.

(This matches above!)

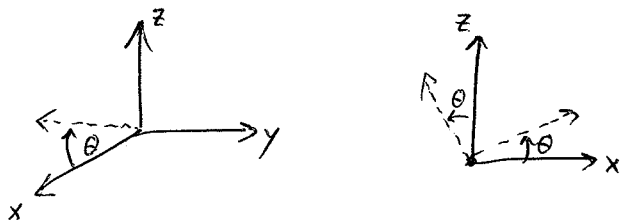
Note $3B = (3 \cdot I_n)B$, so

$$\det(3B) = \det((3 \cdot I_n)B) = \det(3 \cdot I_n) \det B,$$

and $\det(3I_n) = \begin{vmatrix} 3 & & 0 \\ & \ddots & \\ 0 & & 3 \end{vmatrix} = 3^n$, so that $\boxed{\det(3B) = 3^n \det B}$.

4. Let R_θ be the linear transformation that rotates \mathbb{R}^3 about the y -axis by θ radians in the direction taking the positive x -axis toward the positive z -axis.

(a) [4] Find the matrix for R_θ with respect to the standard basis of \mathbb{R}^3 .



The y -component of a vector will be unchanged; we can draw the situation on the xz -plane.

The vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ will be rotated to $R_\theta(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix}$, and

$\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ will rotate to $R_\theta(\vec{e}_3) = \begin{bmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$. As noted, we have $R_\theta(\vec{e}_2) = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Thus, the matrix of R_θ with respect to $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is $\begin{bmatrix} R_\theta(\vec{e}_1) & R_\theta(\vec{e}_2) & R_\theta(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$.

(b) [6] Compute A^{99} where $A = \begin{bmatrix} \sqrt{3} & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & \sqrt{3} \end{bmatrix}$.

[Hint: Think geometrically. Note $\sin(\frac{\pi}{6}) = \frac{1}{2}$. What is $\frac{1}{2}A$?

The hint suggests looking at $\frac{1}{2}A = \begin{bmatrix} \sqrt{3}/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \cos(\pi/6) & 0 & -\sin(\pi/6) \\ 0 & 1 & 0 \\ \sin(\pi/6) & 0 & \cos(\pi/6) \end{bmatrix}$,

which is the matrix of $R_{\pi/6}$ with respect to the standard basis of \mathbb{R}^3 .

Since composition of linear transformations corresponds to multiplication of matrices, we know

that $(\frac{1}{2}A)^{99}$ is the matrix of $(R_{\pi/6})^{99}$ w/resp. to the std. basis of \mathbb{R}^3 .

But 99 compositions of $R_{\pi/6}$ with itself is rotation about the y -axis by an angle of $\frac{\pi}{6} \cdot 99 = \frac{33\pi}{2}$,

equivalent to rotation by $\frac{\pi}{2}$ (since $\frac{33\pi}{2} = \frac{\pi}{2} + 16\pi$, and 16π is a multiple of 2π).

Thus $\frac{1}{2^{99}} \cdot A^{99}$ is the matrix of $R_{\pi/2}$, which is $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Finally, $A^{99} = 2^{99} \cdot \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2^{99} \\ 0 & 2^{99} & 0 \\ 2^{99} & 0 & 0 \end{bmatrix}$.

5. As a reward for this problem, you will find an *explicit formula* for the Fibonacci sequence $a_0, a_1, a_2, a_3, \dots$ defined recursively by $a_0 = 0, a_1 = 1, a_n = a_{n-1} + a_{n-2}$ (so the terms go 0, 1, 1, 2, 3, 5, 8, 13, ...).

(a) [3] Let $\mathbf{x}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$. Circle a matrix A so that $A\mathbf{x}_{n-1} = \mathbf{x}_n$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

($\mathbf{x}_{n-1} = \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix}$, and $\mathbf{x}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a_{n-1} + a_{n-2} \\ a_{n-1} \end{bmatrix}$, so the matrix A must satisfy $A \cdot \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} a_{n-1} + a_{n-2} \\ a_{n-1} \end{bmatrix}$.)

(b) [2] Find the eigenvalues of A .

$$\lambda = \frac{1+\sqrt{5}}{2} \quad \mu = \frac{1-\sqrt{5}}{2}$$

$$\text{Char. poly. of } A = \det \begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{bmatrix} = \lambda(\lambda - 1) - (-1)(-1) = \lambda^2 - \lambda - 1.$$

$$\lambda^2 - \lambda - 1 = 0 \iff \lambda = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

[Problem 5 continued] To correctly answer the remaining questions it is not necessary that you have correctly found λ and μ . You may assume that $\begin{bmatrix} 1 \\ -\mu \end{bmatrix}$ is an eigenvector of A with eigenvalue λ , and $\begin{bmatrix} 1 \\ -\lambda \end{bmatrix}$ is an eigenvector for A with eigenvalue μ . Note that $\mathbf{x}_1 = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) [3] Use the diagonalization idea to solve for \mathbf{x}_n and circle the correct answer.

i. $\mathbf{x}_n = A^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = C^{-1} D^{n-1} C \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix}$, $D = \begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix}$

ii. $\mathbf{x}_n = A^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = C^{-1} D^{n-1} C \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix}$, $D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

iii. $\mathbf{x}_n = A^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = C D^{n-1} C^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix}$, $D = \begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix}$

iv. $\mathbf{x}_n = A^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = C D^{n-1} C^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix}$, $D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

$(A = CDC^{-1}, \text{ so } A^{n-1} = CD^{n-1}C^{-1}.)$

(d) [2] Find the inverse of C and circle the correct answer.

$C^{-1} = \frac{1}{\lambda - \mu} \begin{bmatrix} \lambda & 1 \\ -\mu & -1 \end{bmatrix}$

$C^{-1} = \frac{1}{\lambda - \mu} \begin{bmatrix} -\lambda & -1 \\ \mu & 1 \end{bmatrix}$

$C^{-1} = \frac{1}{\lambda - \mu} \begin{bmatrix} -1 & -\mu \\ 1 & \lambda \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix}^{-1} = \frac{1}{-\lambda - (-\mu)} \begin{bmatrix} -\lambda & -1 \\ \mu & 1 \end{bmatrix} = \frac{1}{\lambda - \mu} \begin{bmatrix} \lambda & 1 \\ -\mu & -1 \end{bmatrix}$

(Can check by multiplying by C , making sure you get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.)

The punch line of this problem, obtained by combining parts (a)-(d), is the formula:

$a_n = \frac{1}{\lambda - \mu} (\lambda^n - \mu^n) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$. (Cool!)

(This formula holds because $\vec{x}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = A^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = CD^{n-1}C^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = CD^{n-1} \cdot \frac{1}{\lambda - \mu} \begin{bmatrix} \lambda & 1 \\ -\mu & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
 $= CD^{n-1} \cdot \begin{bmatrix} \lambda/(\lambda - \mu) \\ -\mu/(\lambda - \mu) \end{bmatrix} = C \cdot \begin{bmatrix} \lambda^{n-1} & 0 \\ 0 & \mu^{n-1} \end{bmatrix} \begin{bmatrix} \lambda/(\lambda - \mu) \\ -\mu/(\lambda - \mu) \end{bmatrix} = C \cdot \begin{bmatrix} \lambda^n/(\lambda - \mu) \\ -\mu^n/(\lambda - \mu) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix} \begin{bmatrix} \lambda^n/(\lambda - \mu) \\ -\mu^n/(\lambda - \mu) \end{bmatrix} = \frac{1}{\lambda - \mu} \begin{bmatrix} \lambda^n - \mu^n \\ \lambda\mu^n - \mu\lambda^n \end{bmatrix}$;
 now just equate the top entries.)

6. Let T_1 and T_2 be the linear transformations that are reflections in \mathbf{R}^3 across the planes V_1 and V_2 respectively, where V_1 is given by the equation $x + y - z = 0$ and V_2 is given by the equation $2x - y + z = 0$.

(a) [1] Find normal vectors \mathbf{n}_1 to V_1 and \mathbf{n}_2 to V_2 . (They do not need to be unit vectors.)

$$\vec{n}_1 = (1, 1, -1)$$

$$\vec{n}_2 = (2, -1, 1).$$

(b) [1] Verify that $\mathbf{n}_1 \in V_2$ and $\mathbf{n}_2 \in V_1$.

$$\text{Check } \vec{n}_1 \in V_2 : 2x - y + z = 2 \cdot 1 - 1 + (-1) = 0 \quad \checkmark$$

$$\text{Check } \vec{n}_2 \in V_1 : x + y - z = 2 + (-1) - 1 = 0 \quad \checkmark$$

(c) [1] Two planes are said to be orthogonal if their normal vectors are orthogonal. Verify that V_1 and V_2 are orthogonal.

$$\vec{n}_1 \cdot \vec{n}_2 = (1, 1, -1) \cdot (2, -1, 1) = 2 - 1 - 1 = 0 \quad \checkmark$$

(But this was clear from parts (a) & (b): We know that $V_1 = \text{span}\{\vec{n}_1\}^\perp$, and since $\vec{n}_2 \in V_1$, it must be that $\vec{n}_2 \perp \vec{n}_1$. But it's also reassuring just to do the dot product.)

[Problem 6 continued]

(d) [3] Find a nonzero vector $\mathbf{n}_3 \in V_1 \cap V_2$.

Method 1: $V_1 \cap V_2 = \left\{ (x, y, z) \mid \begin{array}{l} x+y-z=0 \\ 2x-y+z=0 \end{array} \right\} = N\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix}\right)$
 $= N\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}\right) = \text{span}\{(0, 1, 1)\}$, so take $\boxed{\vec{n}_3 = (0, 1, 1)}$.

Method 2: Since we want $\vec{n}_3 \in V_1$, we need $\vec{n}_3 \perp \vec{n}_1$, and since we want $\vec{n}_3 \in V_2$, we need $\vec{n}_3 \perp \vec{n}_2$. Since we're in \mathbb{R}^3 , there's an easy way to find such an \vec{n}_3 : let $\vec{n}_3 = \vec{n}_1 \times \vec{n}_2$! (We find $\vec{n}_3 = (0, -3, -3)$.)

(e) [2] Find one basis \mathcal{B} of \mathbf{R}^3 consisting of three vectors that are simultaneously eigenvectors of both T_1 and T_2 . (Remember T_1 and T_2 are the reflections across V_1 and V_2 respectively.)

Since T_1 is a reflection across V_1 , we know $T_1(\vec{v}) = \vec{v}$ for $\vec{v} \in V_1$, and $T_1(\vec{w}) = -\vec{w}$ for $\vec{w} \in V_1^\perp$.

(Recall that $T_1(\vec{x}) = \vec{x} - 2(\vec{x} - \text{Proj}_{V_1}(\vec{x})) = -\vec{x} + 2\text{Proj}_{V_1}(\vec{x})$ for any \vec{x} .)

Thus, $T_1(\vec{n}_1) = -\vec{n}_1$, while $T_1(\vec{n}_2) = \vec{n}_2$ and $T_1(\vec{n}_3) = \vec{n}_3$.

Meanwhile, the analogous property of T_2 implies that $T_2(\vec{n}_2) = -\vec{n}_2$, while

$T_2(\vec{n}_1) = \vec{n}_1$ and $T_2(\vec{n}_3) = \vec{n}_3$. Thus, $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ is a set of vectors that are simultaneously eigenvectors for T_1 & T_2 . They form a basis since they are linearly independent (all are mutually orthogonal) and there are 3 of them.

(f) [2] Show that $T_1 \circ T_2 = T_2 \circ T_1$.

The matrix for T_1 w/resp to the basis $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by part (e),

and the matrix for T_2 w/resp. to $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, also by part (e).

Thus, the matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$ are both $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, by multiplying the

two diagonal matrices in the two orders. Since $T_1 \circ T_2$ and $T_2 \circ T_1$ have the same matrix, they are equal.

7. Let \mathcal{B} be the orthonormal basis of \mathbf{R}^3 given in standard coordinates by

$$\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and let $\mathbf{P} : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be the orthogonal projection onto the plane V .

(a) [3] Write down the matrix B for \mathbf{P} with respect to the orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Since $\mathbf{P}(\vec{v}) = \vec{v}$ for $\vec{v} \in V$, we know $\mathbf{P}(\vec{v}_1) = \vec{v}_1$, and $\mathbf{P}(\vec{v}_2) = \vec{v}_2$.

Since $\mathbf{P}(\vec{w}) = \vec{0}$ for $\vec{w} \in V^\perp$, and since $\vec{v}_3 \in V^\perp$ because $\vec{v}_3 \perp \vec{v}_1$ and $\vec{v}_3 \perp \vec{v}_2$, we know $\mathbf{P}(\vec{v}_3) = \vec{0}$.

$$\text{Thus } B = \begin{bmatrix} [\mathbf{P}(\vec{v}_1)]_{\mathcal{B}} & [\mathbf{P}(\vec{v}_2)]_{\mathcal{B}} & [\mathbf{P}(\vec{v}_3)]_{\mathcal{B}} \\ | & | & | \\ | & | & | \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) [3] Write down the change of basis matrix C with $C\mathbf{e}_j = \mathbf{v}_j$ where

$$\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is the standard basis of \mathbf{R}^3 . Write down C^{-1} . [Hint: no computation needed]

$$C = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}, \quad \text{and } \underbrace{C^{-1} = C^T}_{\substack{C \text{ is} \\ \text{orthogonal!}}} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

(c) [4] Find the matrix A for the projection \mathbf{P} with respect to the standard basis of \mathbf{R}^3 .

$$A = CBC^{-1} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 1/3 & 0 \\ 2/3 & -2/3 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 5/9 & 4/9 & -2/9 \\ 4/9 & 5/9 & 2/9 \\ -2/9 & 2/9 & 8/9 \end{bmatrix}.$$

(Note we could also calculate A by taking the matrix $X = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ 2/3 & -2/3 \end{bmatrix}$ of basis vectors for V , and using the fact that $A = X \cdot X^T$.)

9. Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be a differentiable function satisfying:

$$\begin{array}{ll} f(5, 6) = 5 & f(5, 6.2) = 6 \\ f(5.1, 6) = 6.05 & f(5, 6.1) = 5.5 \\ f(5.01, 6) = 5.1005 & f(5, 5.99) = 4.95 \\ f(5.001, 6) = 5.010005 & \end{array}$$

(a) [2] Use all of the above data to give the best value of the partial derivative $f_x(x, y)$ at the point $(x, y) = (5, 6)$.

Since $f_x(5, 6) = \lim_{h \rightarrow 0} \frac{f(5+h, 6) - f(5, 6)}{h}$, we consider the values of the

difference quotient for small h :

h	$\frac{1}{h}(f(5+h, 6) - f(5, 6))$
0.1	$10 \cdot (6.05 - 5) = 10.5$
0.01	$100 \cdot (5.1005 - 5) = 10.05$
0.001	$1000 \cdot (5.010005 - 5) = 10.005$

...So the values appear to be tending toward approximately 10.

(b) [2] Use all of the above data to give the best value of the partial derivative $f_y(x, y)$ at the point $(x, y) = (5, 6)$.

Since $f_y(5, 6) = \lim_{h \rightarrow 0} \frac{f(5, 6+h) - f(5, 6)}{h}$, we consider the difference quotients

for small h :

h	$\frac{1}{h}(f(5, 6+h) - f(5, 6))$
0.2	$5 \cdot (6 - 5) = 5$
0.1	$10 \cdot (5.5 - 5) = 5$
-0.01	$-100 \cdot (4.95 - 5) = 5$

...So the values are very likely tending toward 5.

(c) [6] Give a linear approximation of the function f . Use your approximation to estimate $f(6, 4)$.

$$\begin{aligned} f(x, y) &\approx f(5, 6) + f_x(5, 6) \cdot (x-5) + f_y(5, 6) \cdot (y-6) \\ &= 5 + 10(x-5) + 5(y-6). \end{aligned}$$

$$\text{So } f(6, 4) \approx 5 + 10(6-5) + 5(4-6) = 5 + 10 - 10 = \boxed{5}.$$

10. (a) [5] Let $f(x, y, z) = ax^2 + by^2 + cz^2$ where a , b , and c are constants. Suppose at the point $(-3, 1, 13)$ $f(x, y, z)$ decreases most rapidly in the direction $(6, -7, 5)$. What are the possible values of a , b , and c ?

The given info tells us that $\vec{\nabla} f(-3, 1, 13) = (-6, 7, -5)$, since $\vec{\nabla} f$ points in the direction of most rapid increase of f (and $-\vec{\nabla} f$ thus points in the direction of most rapid decrease).

Now, Since $f = ax^2 + by^2 + cz^2$, we also have $\vec{\nabla} f = (2ax, 2by, 2cz)$, and thus $\vec{\nabla} f(-3, 1, 13) = (-6a, 2b, 26c)$.

Equating $(-6a, 2b, 26c) = (-6, 7, -5)$, we find $a=1, b=\frac{7}{2}, c=-\frac{5}{26}$.

- (b) [5] Suppose $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ and $g : \mathbf{R}^2 \rightarrow \mathbf{R}$ are differentiable with $f(2, 3) = 6$ and $\nabla f(2, 3) = (-1, 4)$, and with $g(2, 3) = 10$ and $\nabla g(2, 3) = (3, 9)$. In what direction at $(2, 3)$ does the product fg increase most rapidly?

We need $\vec{\nabla}(fg)$ at the point $(2, 3)$. One version of the product rule in multiple

variables states $\vec{\nabla}(fg) = g \cdot \vec{\nabla} f + f \cdot \vec{\nabla} g$, so $\vec{\nabla}(fg)(2, 3) = g(2, 3) \cdot \vec{\nabla} f(2, 3) + f(2, 3) \cdot \vec{\nabla} g(2, 3)$
 $= 10 \cdot (-1, 4) + 6 \cdot (3, 9)$
 $= (-10, 40) + (18, 54) = \boxed{(8, 94)}$.

(Alternatively, we seek $\frac{\partial}{\partial x}(fg)$ and $\frac{\partial}{\partial y}(fg)$ at $(2, 3)$,

so by the product rule, $\frac{\partial}{\partial x}(fg)(2, 3) = f_x(2, 3) \cdot g(2, 3) + f(2, 3) \cdot g_x(2, 3) = -1 \cdot 10 + 6 \cdot 3 = 8$,

while $\frac{\partial}{\partial y}(fg)(2, 3) = f_y(2, 3) \cdot g(2, 3) + f(2, 3) \cdot g_y(2, 3) = 4 \cdot 10 + 6 \cdot 9 = 94$.)

11. If $\mathbf{F} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a vector function and you wish to find solutions of $\mathbf{F}(\mathbf{v}) = \mathbf{0}$, Newton's method begins with a first approximation $\mathbf{v}_0 \in \mathbf{R}^n$, then produces a (hopefully) more accurate approximation $\mathbf{v}_1 \in \mathbf{R}^n$ given by

$$\mathbf{v}_1 = \mathbf{v}_0 - (D\mathbf{F}_{\mathbf{v}_0})^{-1}\mathbf{F}(\mathbf{v}_0)$$

where $D\mathbf{F}_{\mathbf{v}_0}$ is the $n \times n$ derivative matrix of \mathbf{F} at \mathbf{v}_0 . Suppose $\mathbf{F} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x^2 + 2y - 2 \\ x^3y - 1 \end{bmatrix}$ and suppose $\mathbf{v}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is a first approximation to a solution of the simultaneous equations $x^2 + 2y - 2 = 0$ and $x^3y - 1 = 0$.

- (a) [3] Find $D\mathbf{F}_{\mathbf{v}_0}$.

$$D\mathbf{F} = \begin{bmatrix} \partial F_1 / \partial x & \partial F_1 / \partial y \\ \partial F_2 / \partial x & \partial F_2 / \partial y \end{bmatrix} = \begin{bmatrix} 2x & 2 \\ 3x^2y & x^3 \end{bmatrix}, \text{ so } D\mathbf{F}_{\vec{v}_0} = \begin{bmatrix} -2 & 2 \\ -3 & -1 \end{bmatrix}.$$

- (b) [3] Find $(D\mathbf{F}_{\mathbf{v}_0})^{-1}$.

$$(D\mathbf{F}_{\vec{v}_0})^{-1} = \begin{bmatrix} -2 & 2 \\ -3 & -1 \end{bmatrix}^{-1} = \frac{1}{(-2)(-1) - (2)(-3)} \begin{bmatrix} -1 & -2 \\ 3 & -2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -1 & -2 \\ 3 & -2 \end{bmatrix}.$$

- (c) [4] Find \mathbf{v}_1 .

$$\begin{aligned} \vec{v}_1 &= \vec{v}_0 - (D\mathbf{F}_{\vec{v}_0})^{-1} \cdot \mathbf{F}(\vec{v}_0) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \frac{1}{8} \begin{bmatrix} -1 & -2 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} (-1)^2 - 2 - 2 \\ (-1)^3(-1) - 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \frac{1}{8} \begin{bmatrix} -1 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \frac{1}{8} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -11/8 \\ 1/8 \end{bmatrix}. \end{aligned}$$

12. (a) [5] Recall that in \mathbf{R}^2 , the relation between polar coordinates (r, θ) and rectangular coordinates (x, y) is given by $x = r \cos \theta$ and $y = r \sin \theta$. If $f(x, y) = x^3 y + y^2 x^2$, express $\left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}\right)$ as a product of two matrices with entries expressed in terms of r and θ . (A "matrix" is allowed to have only one row or column.)

If $g: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is the function defined by $g(r, \theta) = (r \cos \theta, r \sin \theta) (= (x, y))$, then we seek the partial derivatives (w/resp. to r & θ) of the function $f \circ g: \mathbf{R}^2 \rightarrow \mathbf{R}$.

By the chain rule, $\left[\frac{\partial f}{\partial r} \quad \frac{\partial f}{\partial \theta}\right] = D(f \circ g)(r, \theta) = Df(g(r, \theta)) \cdot Dg(r, \theta) = Df(x, y) \cdot Dg(r, \theta)$.

We have $Dg = \begin{bmatrix} \partial g_1 / \partial r & \partial g_1 / \partial \theta \\ \partial g_2 / \partial r & \partial g_2 / \partial \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$ and $Df = \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right] = [3x^2 y + 2y^2 x \quad x^3 + 2yx^2]$,

$$\begin{aligned} \text{so } \left[\frac{\partial f}{\partial r} \quad \frac{\partial f}{\partial \theta}\right] &= [3x^2 y + 2y^2 x \quad x^3 + 2yx^2] \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \\ &= [3r^3 \cos^2 \theta \sin \theta + 2r^3 \sin^2 \theta \cos \theta \quad r^3 \cos^3 \theta + 2r^3 \sin \theta \cos^2 \theta] \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}. \end{aligned}$$

- (b) [5] Suppose $f: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ and $g: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ are differentiable and let $h = g \circ f$ be the composition function. Suppose

$$f(5, 8) = (6, 7) \quad g(6, 7) = (5, 8) \quad f(6, 7) = (5, 8) \quad g(5, 8) = (6, 7)$$

$$\begin{aligned} Dg(6, 7) &= \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} & Dg(5, 8) &= \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \\ Dh(5, 8) &= \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} & Dh(6, 7) &= \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}. \end{aligned}$$

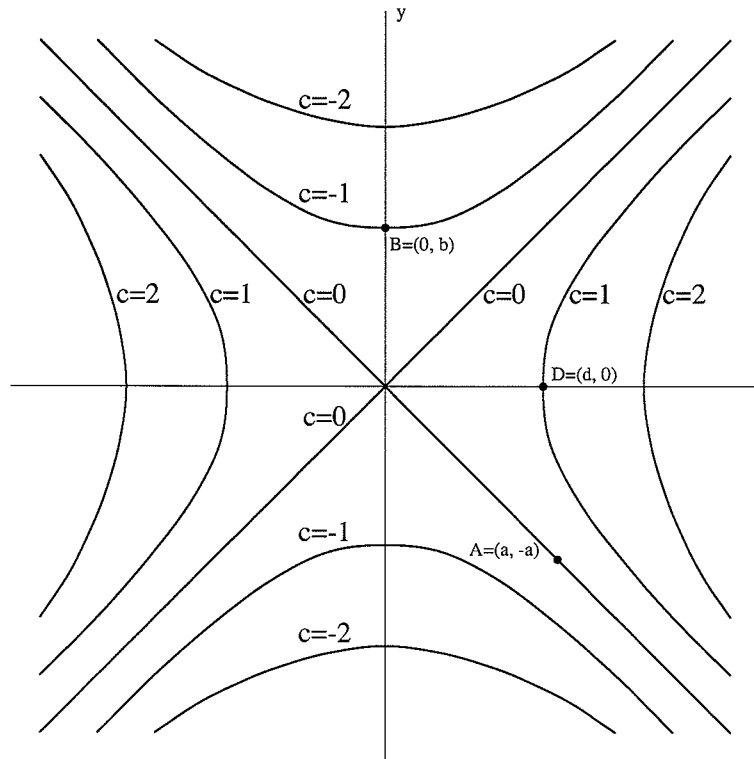
Find $Df(5, 8)$. (WARNING: h is the composition, not f .)

Since $h = g \circ f$, the chain rule implies that

$$\begin{aligned} Dh(5, 8) &= D(g \circ f)(5, 8) = Dg(f(5, 8)) \cdot Df(5, 8) \\ &= Dg(6, 7) \cdot Df(5, 8), \end{aligned}$$

$$\begin{aligned} \text{so } Df(5, 8) &= Dg(6, 7)^{-1} \cdot Dh(5, 8) = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \\ &= \frac{1}{5-2 \cdot 3} \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} = - \begin{bmatrix} -3 & -10 \\ 7 & 23 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 10 \\ -7 & -23 \end{bmatrix}. \end{aligned}$$

13. Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be a differentiable function, and assume that the picture below shows for $c = -2, -1, 0, 1, 2$ the entire level curves $f(x, y) = c$ in the region depicted. Each axis is drawn to the same scale and the positive x and y directions are as usual.



- (a) [2] Circle all possible values of the directional derivative $D_{\mathbf{v}}f$ at the point $A = (a, -a)$ in the direction $\mathbf{v} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

0 1 -1 (1, -1) (-1, 1)

- (b) [2] Circle all possible values of the gradient ∇f of f at the point $(0, 0)$.

(0, 0) (1, 1) (1, -1) (-1, 1) (-1, -1)

- (c) [3] Circle all possible values of the gradient ∇f of f at the point $B = (0, b)$ pictured.

(1, 0) (0, 1) (-1, 0) (0, -1)

- (d) [3] Circle all possible values of the partial derivative f_x at the point $D = (d, 0)$ pictured.

1 -1 5 -7

Part(a): Directional deriv. in the direction of a level curve must be zero (f is const. along level set)

Part(b): $\nabla f(0,0)$ must be orthogonal to level set at this point, but two contour directions force $\nabla f = \vec{0}$.

Part(c): $\nabla f(0,b)$ is orthogonal to the level set and points in direction of increasing f , so only one choice.

Part(d): $f_x = D_{(1,0)}f$ is positive at $(d,0)$, since moving in the pos. x -direction leads to increase in f .

14. Let $f(x, y) = e^{x^2-y}$

(a) [5] Evaluate the Hessian of f at $(1, 1)$.

$$f_x = 2xe^{x^2-y}$$

$$f_y = -e^{x^2-y}$$

$$f_{xx} = 2e^{x^2-y} + 4x^2e^{x^2-y}$$

$$f_{xy} = -2xe^{x^2-y} = f_{yx}$$

$$f_{yy} = e^{x^2-y}$$

$$\Rightarrow Hf(1,1) = \begin{bmatrix} f_{xx}(1,1) & f_{xy}(1,1) \\ f_{xy}(1,1) & f_{yy}(1,1) \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$$

(b) [5] Find the second-order Taylor polynomial of f at $(1, 1)$.

$$\begin{aligned} p_2(1,1) &= f(1,1) + f_x(1,1) \cdot (x-1) + f_y(1,1) \cdot (y-1) \\ &\quad + \frac{1}{2} f_{xx}(1,1) \cdot (x-1)^2 + f_{xy}(1,1) \cdot (x-1)(y-1) + \frac{1}{2} f_{yy}(1,1) \cdot (y-1)^2 \\ &= 1 + 2(x-1) + (-1)(y-1) + 3(x-1)^2 - 2(x-1)(y-1) + \frac{1}{2}(y-1)^2. \end{aligned}$$

15. (a) [4] Find all critical points of the function $f(x, y) = x^2 - y^3 - x^2y + 12y$, that is find all points where $\nabla f = (0, 0)$.

$$f_x = 2x - 2xy,$$

$$f_y = -3y^2 - x^2 + 12. \quad \text{Must set } f_x = f_y = 0.$$

$$\text{Set } f_x = 0 \Rightarrow 2x(1-y) = 0 \Rightarrow x=0 \text{ or } y=1.$$

$$\text{Case } x=0: \text{ Then } f_y = 0 \Rightarrow -3y^2 + 12 = 0 \Rightarrow y^2 - 4 = 0 \Rightarrow y = \pm 2.$$

$$\text{Case } y=1: \text{ Then } f_y = 0 \Rightarrow -x^2 + 9 = 0 \Rightarrow x^2 - 9 = 0 \Rightarrow x = \pm 3.$$

Thus, the critical points are $(0, 2), (0, -2), (3, 1), (-3, 1)$.

- (b) The origin is a critical point of each of the following functions. Classify it as a local max, a local min, or neither.

i. [3] $g(x, y) = x^2 + 4xy + 3y^2$

Have $g_x = 2x + 4y$, $g_y = 4x + 6y$, so $g_{xx} = 2$, $g_{xy} = 4 = g_{yx}$, $g_{yy} = 6$.

Thus Hessian at $(0, 0)$ is $\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}$.

We have $\det \begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix} = 12 - 16 = -4 < 0$,

so the Hessian criterion implies that g has neither a max nor a min at $(0, 0)$. (i.e. a saddle point)

ii. [3] $h(x, y, z) = 3x^2 + 2xy + xz + z^2$

Have $h_x = 6x + 2y + z$, $h_y = 2x$, $h_z = x + 2z$, so $h_{xx} = 6$, $h_{xy} = 2 = h_{yx}$, $h_{xz} = 1 = h_{zx}$,
 $h_{yy} = 0$, $h_{yz} = 0 = h_{zy}$, $h_{zz} = 2$.

Thus Hessian at $(0, 0, 0)$ is $\begin{bmatrix} 6 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Since $h_{xx}(0, 0, 0) = 6 > 0$ while $\det \begin{bmatrix} h_{xx}(0, 0, 0) & h_{xy}(0, 0, 0) \\ h_{xy}(0, 0, 0) & h_{yy}(0, 0, 0) \end{bmatrix} = \det \begin{bmatrix} 6 & 2 \\ 2 & 0 \end{bmatrix} = -4 < 0$, we can already

see that the Hessian criterion implies that h has neither a max nor a min at $(0, 0, 0)$.

16. (a) [5] Maximize the function xy^2z^3 on the part of the plane $x+y+z=6$ with $x \geq 0$, $y \geq 0$, and $z \geq 0$. Explain your reasoning.

The function $f(x,y,z) = xy^2z^3$ won't take on a maximum when $x=0$, or when $y=0$, or when $z=0$, since f can be positive as long as $x>0, y>0, z>0$. So let's restrict our attention to this region and ignore the boundary.

The method of Lagrange Multipliers can be used to identify extrema of f subject to the constraint $g(x,y,z) = x+y+z-6=0$; we must solve the system $\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g=0 \end{cases}$.

We have $\vec{\nabla} f = (f_x, f_y, f_z) = (y^2z^3, 2xy^2z^3, 3xy^2z^2)$ and $\vec{\nabla} g = (1, 1, 1)$, so we have

$$\begin{cases} y^2z^3 = \lambda \\ 2xy^2z^3 = \lambda \\ 3xy^2z^2 = \lambda \\ x+y+z=6 \end{cases} \Rightarrow \text{either } \lambda=0 \text{ or } \frac{\lambda}{1} = \frac{y^2z^3}{2xy^2z^3} = \frac{zy^2z^3}{3xy^2z^2} = \frac{y^2z^3}{3xy^2z^2} \Rightarrow \text{either } \lambda=0 \text{ or } 1 = \frac{y}{2x} = \frac{z}{3x},$$

so either $\lambda=0$ or $6 = x+y+z = x+2x+3x = 6x$. Technical point: Lagrange also requires us to look at case $\vec{\nabla} g = \vec{0}$, but this can't occur.

But $\lambda=0$ implies that one or more of x, y, z is zero, which we can ignore for maxima, so we have $6x=6$, meaning $x=1$, and $y=2x=2$, and $z=3x=3$. The value of f , which must be a max, is $1 \cdot 2^2 \cdot 3^3 = \boxed{108}$.

- (b) [5] Find the maximum and minimum values of the function e^{x^2-y} on the unit circle $x^2+y^2=1$.

Let $f(x,y) = e^{x^2-y}$ and $g(x,y) = x^2+y^2-1$.

Using the method of Lagrange Multipliers, we solve the system $\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g=0 \end{cases}$:

We have $\vec{\nabla} f = (f_x, f_y) = (2xe^{x^2-y}, -e^{x^2-y})$ and $\vec{\nabla} g = (2x, 2y)$, so we have

$$\begin{cases} 2xe^{x^2-y} = 2\lambda x \\ -e^{x^2-y} = 2\lambda y \\ x^2+y^2-1=0 \end{cases} \Rightarrow \frac{2xe^{x^2-y}}{-e^{x^2-y}} = \frac{2\lambda x}{2\lambda y} \Rightarrow \frac{2x}{-1} = \frac{x}{y} \Rightarrow 2xy+x=0 \Rightarrow x=0 \text{ or } y=-\frac{1}{2}.$$

(Note $\lambda \neq 0$)

If $x=0$, we have candidate points $(0,1)$ and $(0,-1)$ on the circle;

if $y=-\frac{1}{2}$, we have candidate points $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ on the circle.

Testing f at each of these points, we find $f(0,1) = \frac{1}{e}$; $f(0,-1) = e$; $f(\frac{\sqrt{3}}{2}, -\frac{1}{2}) = e^{5/4}$; $f(-\frac{\sqrt{3}}{2}, -\frac{1}{2}) = e^{5/4}$.

Thus, the minimum value is $\boxed{\frac{1}{e}}$, and the maximum is $\boxed{e^{5/4}}$. Technical point: Lagrange also requires us to check pts on circle where $\vec{\nabla} g = \vec{0}$, but there are no points where this is true.