

MATH 51 MIDTERM II

November 17, 2005

Name: _____

Numeric Student ID: _____

Instructor's Name: _____

I agree to abide by the terms of the honor code:

Signature: _____

Instructions: Print your name, student ID number and instructor's name in the space provided. During the test you may not use notes, books or calculators. Read each question carefully and **show all your work**; full credit cannot be obtained without sufficient justification for your answer unless explicitly stated otherwise. Underline your final answer to each question. There are 9 questions. You have 90 minutes to do all the problems.

Question	Score	Maximum
1		10
2		6
3		8
4		6
5		10
6		10
7		15
8		10
9		10
Total		85

1. Consider the function

$$f(x, y) = x^4y^3$$

and the point $P = (1, 1, 1)$ on its graph.

- (a) Write down the equation of the tangent plane at the graph of the function at the point P .

Solution:

The tangent plane has equation, in general,

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right\rangle \Big|_P \cdot (\mathbf{x} - P) = 0$$

In this case, $\frac{\partial f}{\partial x} = 4x^3y^3$ which is 4 at $(1, 1)$. $\frac{\partial f}{\partial y} = 3x^4y^2$ which is 3 at $(1, 1)$. So the normal vector to the plane is $\langle 4, 3, -1 \rangle$ and the tangent plane has equation:

$$4(x - 1) + 3(y - 1) - (z - 1) = 0$$

- (b) Using your answer from (a), write down an expression for the change, Δz , in $z = f(x, y)$ depending on Δx and Δy , the change in x and y , respectively, near the point $P = (1, 1, 1)$. Is the function $f(x, y)$ more sensitive to a change in x or to a change in y ? Explain.

Solution:

Near the point $P = (1, 1, 1)$ the tangent plane is a good approximation to the function. Thus

$$\Delta z = 4\Delta x + 3\Delta y$$

hence the function is more sensitive to changes in x . That is, small changes in x result in larger differences in the resulting z than small changes in y .

- (c) Using your answer to (b), find the approximate value of $f(1.01, 1.02)$.

Solution:

Using the tangent plane approximation at $(1, 1)$, we find

$$f(1.01, 1.02) = f(1, 1) + .01(4) + .02(3) = 1 + .04 + .06 = 1.1$$

since we know using any linear approximation in two variables, this is just the value of the tangent plane at this point.

2. (a) The steady state temperature function $T(x, y)$ for a thin flat plate satisfies the equation

$$T_{xx} + T_{yy} = 0.$$

Does the function

$$T(x, y) = \ln(x^2 + y^2)$$

satisfy the equation above? Show your work.

Solution:

T_{xx} denotes the second partial derivative with respect to x .

$$T_x = \frac{\partial T}{\partial x} = \frac{2x}{x^2 + y^2}$$

using the chain rule. Then using the quotient rule,

$$T_{xx} = \frac{2(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

The function is symmetric in x and y so the derivatives in y will be the same with the roles reversed. Substituting, we see that the function does indeed satisfy the above partial differential equation.

(b) Given

$$f(x, y, z) = z \sin x + \frac{\cos y \ln(z + y)}{z},$$

compute the partial derivative $\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial z} \right)$.

Solution:

One can take this in the order suggested, but then it is a bit messy to take the first partial derivative with respect to z . Instead, it's better to remember that the mixed partial derivatives of a function are equal wherever the mixed partials exist and are continuous. Then taking the partial derivative with respect to x first, we have

$$\frac{\partial f}{\partial x} = z \cos x$$

so that

$$\frac{\partial^2 f}{\partial z \partial x} = \cos x$$

3. (a) Compute the determinant of the matrix

$$\begin{pmatrix} 5 & 11 \\ -2 & 3 \end{pmatrix}.$$

Solution:

The determinant is $5 \cdot 3 - (-2) \cdot 11 = 37$.

- (b) What are the values of the parameter a for which the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & a+2 \\ 1 & 4-a & 5 \end{pmatrix}$$

is invertible?

Solution:

The determinant is $a^2 - a - 2 = (a+1)(a-2)$. Therefore, the matrix is invertible for all $a \in \mathbb{R}$ except $\{2, -1\}$.

4. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}.$$

Solution:

We use the augmented matrix with the identity and do Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 3 & 3 & 3 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 6 & 0 & -3 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 6 & 0 & -3 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 0 & 1/6 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 1/6 \\ 0 & 1 & 0 & -1/2 & 0 & 1/6 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & 1/6 \\ 0 & 1 & 0 & -1/2 & 0 & 1/6 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 0 & -1/2 & 1/6 \\ -1/2 & 0 & 1/6 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

5. (a) The linear operator $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by

$$T(x_1, x_2) = (3x_1 + 4x_2, x_1 + 3x_2).$$

Find a basis of \mathbb{R}^2 such that the matrix of T with respect to that basis is diagonal.

Solution:

The matrix A of T with respect to the standard basis and its characteristic polynomial p_A are:

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}, \quad p_A(\lambda) = \det(A - \lambda I) = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5).$$

There are two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$. Let us find the corresponding eigenvectors.

$\lambda_1 = 1$: we need to find the solutions of the homogeneous linear system corresponding to the matrix $A - I$.

$$A - I = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad x_1 = -2x_2.$$

The eigenvectors are $\text{span}\{v_1\}$, where $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$\lambda_2 = 5$: we need to find the solutions of the homogeneous linear system corresponding to the matrix $A - 5I$.

$$A - 5I = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \quad x_1 = 2x_2.$$

The eigenvectors are $\text{span}\{v_2\}$, where $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

The matrix of T with respect to the basis $\{v_1, v_2\}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$.

(b) Are there real numbers a and b such that the matrices

$$A = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

are similar (i.e. $A = CBC^{-1}$)? If yes, find a and b . If no, explain why not.

Solution:

If A and B were similar then the matrix A should have two linearly independent eigenvectors with eigenvalue 1 and one eigenvector with eigenvalue -2 .

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & a \\ 1 & -\lambda & b \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda^3 + b\lambda + a.$$

Since 1 and -2 are eigenvalues we get

$$\begin{cases} b + a = 1 \\ -2b + a = -8 \end{cases}, \quad 3b = 9, \quad b = 3 \text{ and } a = -2.$$

Let us find the eigenvectors with eigenvalue 1:

$$A - I = \begin{bmatrix} -1 & 0 & -2 \\ 1 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = -2x_3, \quad x_2 = x_3$$

We get that the space of eigenvectors with eigenvalue 1 is $\text{span}\{v\}$, where $v =$

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \text{ it is 1 (not 2) dimensional, thus the answer to the question is no.}$$

6. Determine the definiteness (positive/negative definite or semidefinite, indefinite) of the following quadratic forms. Explain your answers.

(a) The quadratic form in two variables,

$$Q(x,y) = x^2 + 6xy + 2y^2.$$

Solution:

The quadratic form is positive definite, since the determinant and the trace are both positive numbers.

(b) The quadratic form associated to the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Solution:

The characteristic polynomial is $(\lambda-1)(\lambda+1)(\lambda-3)$. The matrix has three distinct eigenvalues and therefore is diagonalizable. Since not all eigenvalues are positive numbers or negative numbers, the associated quadratic form is indefinite.

7. Determine the matrix associated to the following linear transformations. Express your answers in terms of the standard basis.

(a) The transformation $T(x_1, x_2) = (7x_1 - x_2, 8x_1 + 3x_2)$ where $(x_1, x_2) \in \mathbb{R}^2$.

Solution:

The matrix associated to a linear transformation with respect to the standard basis has columns $T(e_1)$ and $T(e_2)$. Now $T(e_1) = (7, 8)$ and $T(e_2) = (-1, 3)$. So writing these in a matrix we have:

$$T \leftrightarrow A = \begin{pmatrix} 7 & -1 \\ 8 & 3 \end{pmatrix}$$

(b) Rotation in \mathbb{R}^2 by an angle of $\pi/3$ radians.

Solution:

Again, the matrix is determined by its action on basis vectors. For the rotation,

$$T(e_1) = \begin{pmatrix} \cos \pi/3 \\ \sin \pi/3 \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} -\sin \pi/3 \\ \cos \pi/3 \end{pmatrix}$$

so the associated matrix in the standard basis is:

$$\begin{pmatrix} \cos \pi/3 & -\sin \pi/3 \\ \sin \pi/3 & \cos \pi/3 \end{pmatrix}$$

(c) Projection of vectors in \mathbb{R}^3 onto the line L given by the points

$$\left\{ \left[\begin{array}{c} 3t \\ -2t \\ t \end{array} \right] \mid t \in \mathbb{R} \right\}.$$

Solution:

Projection of vectors onto a line, in any number of dimensions, is given by the formula

$$\text{Proj}_u(v) = \frac{u \cdot v}{\|v\|} u$$

since in any number of dimensions, e.g. 3, the projection is given by forming the right triangle with hypotenuse v and angle θ , the angle between u and v in the plane where they both lie. The direction vector for the line, scaled to be a unit vector, is

$$u = \frac{1}{\sqrt{14}} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

Again we look at the action of the projection on the standard basis. These vectors form the columns of the associated matrix for the transformation.

$$T(e_1) = \begin{pmatrix} 9/14 \\ -6/14 \\ 3/14 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} -6/14 \\ 4/14 \\ -2/14 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} 3/14 \\ -2/14 \\ 1/14 \end{pmatrix}$$

with associated matrix:

$$\begin{pmatrix} 9/14 & -3/7 & 3/14 \\ -3/7 & 2/7 & -1/7 \\ 3/14 & -1/7 & 1/14 \end{pmatrix}.$$

(d) Reflection in \mathbb{R}^3 in the plane given by

$$3y - z = 0.$$

Solution:

This transformation is hard to describe in terms of the standard basis, which is ill-suited for describing reflections. A better approach is to choose a nice basis for the reflection and then change back to the standard basis. For the reflection, we choose two vectors in the plane of reflection, which will stay fixed, and the normal vector for the plane, which will go to the negative of itself. That is, choose basis $\{v_1, v_2, v_3\}$ with

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

Then $T(v_1) = v_1$, $T(v_2) = v_2$ and $T(v_3) = -v_3$ so the matrix corresponding to this transformation in the basis made of v_i is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The change of basis matrix is

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & -1 \end{pmatrix}$$

The inverse of this change of basis matrix is:

$$C^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/10 & 3/10 \\ 0 & 3/10 & -1/10 \end{pmatrix}$$

which can be computed immediately from the 2×2 formula since the matrix has first row and first column as in the identity matrix. Finally, the transformation is given by

$$CBC^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4/5 & 3/5 \\ 0 & 3/5 & 4/5 \end{pmatrix}.$$

8. Let V be the 2-dimensional subspace of \mathbb{R}^3 with basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix} \right\}.$$

(a) Given the vector in \mathcal{B} -coordinates

$$\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{\mathcal{B}},$$

express \mathbf{v} in standard coordinates.

Solution:

To change from \mathcal{B} coordinates to standard coordinates, we simply multiply by the change of basis matrix:

$$\begin{pmatrix} 3 & 2 \\ -5 & 0 \\ 1 & 9 \end{pmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 16 \\ -20 \\ 22 \end{bmatrix}$$

(b) Express the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

written in standard coordinates in terms of the \mathcal{B} -coordinates.

Solution:

We can't invert the change of basis matrix as it isn't square, for one. Instead we go back to the definition of coordinates, realizing we have to write

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}$$

From the second row, $-5c_1 = -1$ so $c_1 = 1/5$. Then from the first row,

$$1 = 3c_1 + 2c_2 = 3/5 + 2c_2$$

and $c_2 = 1/5$, so

$$v = \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix}_{\mathcal{B}}$$

9. Let A and B be two invertible $n \times n$ matrices. Show that AB and BA have the same characteristic polynomial.

Solution:

Recall that any two SIMILAR matrices have the same characteristic polynomial. So we will prove the claim if we can exhibit an invertible matrix C such that $CABC^{-1} = BA$. Choosing $C = A^{-1}$, which is valid since A is invertible, gives the desired equality.