MATH 51 MIDTERM

October 20, 2005

Name: ______________________________

Numeric Student ID: ____________________

Instructor's Name: _____________________

I agree to abide by the terms of the honor code:

Signature: ___________________________

Instructions: Print your name, student ID number and instructor's name in the space provided. During the test you may not use notes, books or calculators. Read each question carefully and **show all your work**; full credit cannot be obtained without sufficient justification for your answer unless explicitly stated otherwise. Underline your final answer to each question. There are **??** questions. You have 90 minutes to do all the problems.

Question	Score	Maximum
1		10
2		15
3		10
4		10
5		10
6		10
7		10
8		10
Total		85

1. Solve the following system of equations in the variables x, y, z, w:

$$\begin{aligned} x - y - z + w &= 5 \\ y - z + 2w &= 8 \\ 2x - y - 3z + 4w &= 18 \end{aligned}$$

If a solution exists, express your answer in parametric form.

Solution:

Writing this in matrix form, we obtain

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 5 \\ 0 & 1 & -1 & 2 & 8 \\ 2 & -1 & -3 & 4 & 18 \end{array}\right).$$

We will solve this system of equations by performing Gaussian elimination. We proceed as follows. First, subtract twice the top row from the third row. This operation yields

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 5 \\ 0 & 1 & -1 & 2 & 8 \\ 0 & 1 & -1 & 2 & 8 \end{array}\right).$$

Now subtract the second row from the third row, which yields

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 5 \\ 0 & 1 & -1 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

At this point, we can read off the solution. First, the variables z and w are free. Next, $y = z - 2w$ and $x = (z - 2w) + z - w = 2z - 3w$. Expressed parametrically, this is

$$\left\{ \begin{pmatrix} 2z - 3w \\ z - 2w \\ z \\ w \end{pmatrix} \middle| \, z, w \in \mathbb{R} \right\}$$

or

$$\left\{ z \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \middle| \, z, w \in \mathbb{R} \right\}.$$

2. Let A be the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{pmatrix}.$$

(a) Find a basis for the nullspace of A.

Solution:

For (a), we transform the matrix A to its RRE form:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \mapsto$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are therefore two free variables x_3 and x_5 and three fixed variables x_1, x_2 and x_4. The homogeneous linear system associated to the RRE form is then:

$$x_1 - x_3 + x_5 = 0$$

$$x_2 + x_3 + 2x_5 = 0$$

$$x_4 + x_5 = 0.$$

Solving this, one obtains:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_3 - x_5 \\ -x_3 - 2x_5 \\ x_3 \\ -x_5 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

The null space of A is then spanned by the two vectors:

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

One verifies that these vectors are linearly independent. Hence, they form a basis for $N(A)$.

(b) Find a basis for the column space of A.

Solution:

For (b), note that the columns containing pivots in the RRE form of A are the 1st, 2nd and 4th. Therefore, 1st, 2nd and 4th columns of A give vectors in R^4 that will form a basis for $C(A)$. To justify this step recall the procedure presented in class.

(c) Using your work from the previous page, what is the set of all solutions to the equation

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}?$$

State your answer in parametric form.

Solution:

For (c), note that:

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

Therefore a solution for

$$Ax = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$

always has the form:

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

3. Let P_1 be the plane described by normal vector $(1, 1, 1)$ and containing the point $(0, 0, 1)$. Let P_2 be the plane described by the equation $x + 2y + 3z = 0$.

   (a) Write P_1 as an equation of the form $ax + by + cz = d$.

   (b) What is the set of points in the intersection of P_1 and P_2? If there are points in the intersection, express them in parametric form.

Solution:

(a) Recall that the equation describing a plane in terms of a normal vector n and a displacement point p is $n \cdot (x - p) = 0$. Plugging in the given values, we find that the left side of the equality is $(1, 1, 1) \cdot ((x, y, z) - (0, 0, 1))$, and expanding this out we have $(1, 1, 1) \cdot (x, y, z - 1) = x + y + z - 1$. Therefore, the plane admits an alternate description as the solutions to the equation $x + y + z = 1$.

(b) The set of points in the intersection of the two planes must simultaneously satisfy the equations $x + y + z = 1$ and $x + 2y + 3z = 0$. Writing this in matrix form, we have

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{array}\right).$$

As usual, we solve by performing Gaussian elimination. Subtracting the top row from the bottom, we obtain

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

Then subtracting the new bottom row from the top, we obtain

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

Therefore z is a free variable, $y = -2z - 1$, and $x = z + 2$. Expressed parametrically, the solutions are

$$\left\{ \left(\begin{array}{c} z + 2 \\ -2z - 1 \\ z \end{array} \right) \middle| z \in \mathbb{R} \right\}$$

or

$$\left\{ z \left(\begin{array}{c} 1 \\ -2 \\ 1 \end{array} \right) + \left(\begin{array}{c} 2 \\ -1 \\ 0 \end{array} \right) \middle| z \in \mathbb{R} \right\}.$$

4. The coordinates of three points P, Q and R are $(1, 1, 1)$, $(2, 1, 0)$ and $(3, 2, 3)$ respectively.

(a) Show that the vectors $\vec{PQ}$ and $\vec{PR}$ are perpendicular.

Solution:

We have

$$\overrightarrow{PQ}=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \overrightarrow{PR}=\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

To show that $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are perpendicular, we compute the dot product

$$\overrightarrow{PQ} \cdot \overrightarrow{PR}= 1 \cdot 2 + 0 \cdot 1 + (-1) \cdot 2 = 0.$$

(b) Find the equation of the plane passing through the points P, Q and R. Express your answer in terms of variables x, y, z in $\mathbb{R}^3$.

Solution:

A normal vector to the plane through P, Q and R is given by the cross product

$$\overrightarrow{PQ} \times \overrightarrow{PR}=\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}.$$

The equation of the plane is found by requiring

$$\begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = 0 \Leftrightarrow x - 4y + z = -2.$$

(c) Determine the area of the triangle P, Q and R.

Solution:

The area of the triangle with vertices P, Q, R is given by

$$\frac{1}{2} || \overrightarrow{PQ} \times \overrightarrow{PR} || = \frac{1}{2}\sqrt{1^2 + (-4)^2 + 1^2} = \frac{3\sqrt{2}}{2}.$$

5. For each of the following sets, determine whether or not the set is a subspace. For this question only, you do not need to show your work; simply write SUBSPACE or NOT SUBSPACE.

(a) The set $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ in } \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 1 \right\}$.

Solution:
NOT A SUBSPACE (doesn't contain 0 vector, not preserved under addition or scalar multiplication)

(b) The set $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ in } \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n \right\}$.

Solution:
NOT A SUBSPACE (not preserved under scalar mult. by negative numbers, intuitively not a linear object either (a line, plane, etc))

(c) The nullspace $N(A)$, where

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 6 & -17 \end{pmatrix}.$$

Solution:
SUBSPACE (sort of a trick question: the null space is always a subspace, no matter what the matrix A is)

(d) The set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = x^2 \right\}$.

Solution:
NOT A SUBSPACE (not preserved under addition or scalar multiplication, and again, intuitively its a parabola, not a line or plane)

(e) The set $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ in } \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 0 \right\}$.

Solution:
SUBSPACE (you can check all three properties directly. Intuitively, this is a space of one less dimension than the total space. So if $n = 3$, this one equation gives a plane.)

6. Let $\{u, v, w\}$ be a linearly independent set of vectors. Show that the set

$$\{u, u + 2v, u + 2v + 3w\}$$

is a linearly independent set of vectors, as well.

Solution:

We use the standard criterion (Proposition 3.1 in the textbook) to check if $\{u, u + 2v, u + 2v + 3w\}$ is a linearly independent set. We look therefore at the equation:

$$au + b(u + 2v) + c(u + 2v + 3w) \;=\; 0.$$

This can be rewritten as:

$$(a + b + c)u + (2b + 2c)v + 3cw = 0.$$

Since we know that $\{u, v, w\}$ is a linearly independent set, it has to be that:

$$a + b + c = 0, \quad 2b + 2c = 0, \quad 3c = 0.$$

It follows from here that $a = b = c = 0$ and therefore one concludes that $\{u, u + 2v, u + 2v + 3w\}$ is linearly independent.

7. Let

$$A = \begin{pmatrix} 5 & -2 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Find all solutions to the equation $A\mathbf{x} = 3\mathbf{x}$.

Solution:

There are two ways to solve this problem, essentially the same. First, write down a system of equations corresponding to $A\mathbf{x} = 3\mathbf{x}$ in the variables x_1, x_2, x_3 and then rewrite your system so that you are solving a system with each equation equal to 0.

Second, you could think of these as linear transformations. Then $3\mathbf{x}$ should be regarded as corresponding to the matrix

$$3I_3 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

so that we can subtract linear transformations and solve $(A - 3I_3)\mathbf{x} = 0$. The matrix for $(A - 3I_3)$ is

$$A - 3I_3 = \begin{pmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}$$

The RREF$(A - 3I_3)$ is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

so our answer is the span of the vector $\mathbf{v} = [1, 1, 1]$. To check our answer, we note that $A\mathbf{v} = 3\mathbf{v}$, and we know how linear transformations behave under multiplication by scalars, so all our solutions clearly satisfy the original equation.

8. Let V be the set of all points P in $\mathbb{R}^4$ such that the distance from P to each one of the points $(1, 2, 3, 4)$, $(2, 3, 4, 1)$, and $(3, 4, 2, 1)$ are all equal. Show that V is a linear subspace of $\mathbb{R}^4$ and compute its dimension.

   Note: The distance between two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is by definition:

$$\text{dist}(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}.$$

Solution:

Let V be the set of all points A in $\mathbb{R}^4$ such that the three distances from A to each one of the points $(1, 2, 3, 4)$, $(2, 3, 4, 1)$, and $(3, 4, 2, 1)$ are equal. Show that V is a linear subspace of $\mathbb{R}^4$ and compute its dimension.

Note: The distance between two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is by definition: $\text{dist}(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$.

Solution. Put

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 1 \end{bmatrix}.$$

A vector x in $\mathbb{R}^4$ belongs to the set V if and only if

$$\|x - a\|^2 = \|x - b\|^2 = \|x - c\|^2$$
$$x \cdot x - 2x \cdot a + a \cdot a = x \cdot x - 2x \cdot b + b \cdot b = x \cdot x - 2x \cdot c + c \cdot c$$

Note that $a \cdot a = b \cdot b = c \cdot c = 1^2 + 2^2 + 3^2 + 4^2$. Thus the above equalities are equivalent to the system of linear equations

$$(b - a) \cdot x = 0 \qquad x_1 + x_2 + x_3 - 3x_4 = 0$$
$$(c - b) \cdot x = 0 \qquad x_1 + x_2 - 2x_3 = 0$$

The subset V is a linear subspace because it coincides with the null space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

The first and the third columns of A are linearly independent, thus $\dim C(A) = 2$. Since $\dim C(A) + \dim N(A) = 4$, we must have $\dim V = \dim N(A) = 2$.