MATH 51 MIDTERM 2 SOLUTIONS

February 26, 2004

Professors Clingher, Munson, and White

1. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. **Solution:**

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & .5 & .5 \\ 0 & 0 & 1 & .5 & .5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & .5 & 0 & .5 \\ 0 & 1 & 0 & 0 & .5 & .5 \\ 0 & 0 & 1 & .5 & .5 & 0 \end{array}\right]$$

so

$$\boxed{A^{-1} = \begin{bmatrix} .5 & 0 & .5 \\ 0 & .5 & .5 \\ .5 & .5 & 0 \end{bmatrix}}.$$

2. Suppose A and B are 3x3 matrices, and that $\det(A) = 5$ and $\det(B) = -2$.

(a) Find $\det(AB)$. **Solution:** $\det(AB) = \det(A)\det(B) = 5(-2) = \boxed{-10}$.

(b) Find $\det(A^{-1})$. **Solution:** $\det(A^{-1}) = (\det A)^{-1} = \boxed{1/5}$.

(c) Find $\det(2A)$. **Solution:** $\det(2A) = 2^3 \det(A) = 8(5) = \boxed{40}$.

3. Let $A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$.

(a) Find the eigenvalues of A. **Solution:**

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 1 \\ -2 & \lambda - 3 \end{vmatrix} = \lambda(\lambda - 3) - 1(-2) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2),$$

so the eigenvalues are $\boxed{1}$ and $\boxed{2}$.

(b) Find the eigenvalues of A^{10}. **Solution:** $\boxed{1^{10} = 1}$ and $\boxed{2^{10}}$ (or 1024).

(c) The matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

has the number 3 as one of its eigenvalues. Find an eigenvector $\mathbf{v}$ that has 3 as its associated eigenvalue.

Solution: We need a nonzero vector in the nullspace of $3I - A$. We find it by Gaussian elimination:

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\mathbf{v}$ is in the nullspace of $3I - A$ if and only if $v_1 - 2v_3 = 0$ and $v_2 - v_3 = 0$, i.e., if and only if

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} v_3.$$

So $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (or any nonzero multiple of it) is an eigenvector of A with eigenvalue 3.

4. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation defined by:

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ -2x + 4y \end{bmatrix}.$$

(a). Find the matrix A that represents the linear transformation T with respect to the standard basis $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2\}$.

SOLUTION: $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, so

$$\boxed{A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}}.$$

(b). Consider the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ given by:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Find the change of basis matrix C for the basis $\mathcal{B}$. That is, find the matrix C such that $\mathbf{v} = C[\mathbf{v}]_{\mathcal{B}}$ for all vectors $\mathbf{v}$.

Solution: The columns of C are the basis vectors $\mathbf{v}_1$ and $\mathbf{v}_2$: $\boxed{C = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}$.

(c). Find the matrix B that represents the linear transformation T with respect to basis $\mathcal{B}$.

Solution: $B = C^{-1}AC$, so we need C^{-1}:

$$\left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc:cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

so $C^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$. Thus

$$\boxed{B = C^{-1}AC = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}.}$$

5. Find each of the following limits, or else explain clearly why the limit does not exist.

(a). $\lim_{(x,y)\to(2,3)} \dfrac{e^{xy}\sin y}{2x+y}$.

Solution: The function is continuous at $(2,3)$, so the limit is $\dfrac{e^{2\cdot 3}\sin 3}{2(2)+3} = \boxed{\dfrac{e^6 \sin 3}{7}}$.

(b). $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$.

Solution: In polar coordinates, the expression is

$$\frac{(r\cos\theta)(r\sin\theta)}{r^2} = \cos\theta\sin\theta$$

Thus the function is constant on each ray coming from the origin. But the constant depends on the ray, so the limit does not exist.

(c). $\lim_{(x,y)\to(0,0)} \dfrac{xy\sin x}{x^2+2y^2}$.

Solution: In polar coordinates, the expression is

$$\begin{aligned} \frac{(r\cos\theta)(r\sin\theta)\sin(r\cos\theta)}{r^2\cos^2\theta + 2r^2\sin^2\theta} &= \frac{(\cos\theta)(\sin\theta)\sin(r\cos\theta)}{\cos^2\theta + 2\sin^2\theta} \\ (*) \qquad\qquad &= \frac{(\cos\theta)(\sin\theta)\sin(r\cos\theta)}{1+\sin^2\theta}. \end{aligned}$$

If $r \to 0$, then $r\cos\theta \to 0$, so $\sin(r\cos\theta) \to \sin(0) = 0$ (because sin is continuous), so the numerator of (*) goes to 0. The denominator of (*) is always ≥ 1 and ≤ 2, so (*) $\to 0$. Thus the limit is 0.

6(a). Find a matrix A such that

$$A\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},\qquad A\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix},\qquad A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}7\\7\end{bmatrix}.$$

Solution: The first equation tells us that $\begin{bmatrix}1\\2\end{bmatrix}$ is the first column of A. Also,

$$A\mathbf{e}_2 = A\left(\begin{bmatrix}1\\1\\0\end{bmatrix}-\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=A\begin{bmatrix}1\\1\\0\end{bmatrix}-A\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}-\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$$

so the second column is $\begin{bmatrix}3\\1\end{bmatrix}$. Finally

$$A\mathbf{e}_3 = A\left(\begin{bmatrix}1\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=A\begin{bmatrix}1\\1\\1\end{bmatrix}-A\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}7\\7\end{bmatrix}-\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$$

so the third column is $\begin{bmatrix}3\\4\end{bmatrix}$. Thus

$$\boxed{A=\begin{bmatrix}1&3&3\\2&1&4\end{bmatrix}}$$

(b). Let $T:\mathbf{R}^2\to\mathbf{R}^2$ be reflection across the line $y=3x$. Find the matrix for T (with respect to the standard basis of $\mathbf{R}^2$.)

Solution: Pick a nonzero vector in the line, say $(1,3)$. Then (using the formula in the book) the projection matrix is

$$P=\frac{1}{1^2+3^2}\begin{bmatrix}1\cdot1&1\cdot3\\3\cdot1&3\cdot3\end{bmatrix}=\frac{1}{10}\begin{bmatrix}1&3\\3&9\end{bmatrix}$$

As explained in the text, the reflection matrix is

$$R=2P-I=\frac{1}{5}\begin{bmatrix}1&3\\3&9\end{bmatrix}-I=\boxed{\begin{bmatrix}-4/5&3/5\\3/5&4/5\end{bmatrix}}.$$

Another method: Let $\mathbf{v}_1$ be a vector in the line, say $(1,3)$. Let $\mathbf{v}_2$ be a vector perpendicular to the line, say $(-3,1)$. The reflection takes $\mathbf{v}_1$ to $\mathbf{v}_1$ and $\mathbf{v}_2$ to $-\mathbf{v}_2$, so the matrix B for this reflection with respect to the basis $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2\}$ is:

$$B=\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

If A is the matrix for T in with respect to the standard basis, then $B = C^{-1}AC$, where C is the matrix with columns $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus

$$A = CBC^{-1} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}}.$$

7. A particle moves through space with velocity given by $\mathbf{v}(t) = (\sin t, 2t, 1)$. At time $t = 0$, the particle's position is $(2, 3, 4)$.

(a) Find the particle's speed at time t. **Solution:** $\|\mathbf{v}(t)\| = \sqrt{(\sin t)^2 + (2t)^2 + 1^2} = \boxed{\sqrt{\sin^2 t + 4t^2 + 1}}$.

(b) Find the particle's acceleration vector at time t. **Solution:** $\boxed{\mathbf{v}'(t) = (\cos t, 2, 0)}$.

(c) Find the particle's position $\mathbf{x}(t)$ at time t. **Solution:**

$$\mathbf{x}(t) = \int \mathbf{v}(t)\, dt = \int (\sin t, 2t, 1)\, dt = (-\cos t, t^2, t) + \mathbf{C}.$$

Setting $t = 0$ gives

$$(2, 3, 4) = \mathbf{x}(0) = (-\cos 0, 0^2, 0) + \mathbf{C} = (-1, 0, 0) + \mathbf{C}$$

so $\mathbf{C} = (3, 3, 4)$. Thus

$$\boxed{\mathbf{x}(t) = (-\cos t + 3, t^2 + 3, t + 4)}.$$

8. Calculate the following partial derivatives:

(a) $\dfrac{\partial}{\partial x}(xy^2z + y^2 \sin x + yz^5)$. **Solution:** $\boxed{y^2z + y^2 \cos x}$.

(b) $\dfrac{\partial}{\partial y} \sin(xyz + x^2)$. **Solution**: $\cos(xyz + x^2)\dfrac{\partial}{\partial y}(xyz + x^2) = \boxed{\cos(xyz + x^2)xz}$.

(c) $\dfrac{\partial f}{\partial x}(2, 3)$ where $f(x, y) = x^2 + xy + y^2$. **Solution:** $\dfrac{\partial f}{\partial x} = 2x + y$, so $\dfrac{\partial f}{\partial x}(2, 3) = 2(2) + 3 = \boxed{7}$.

(d) $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$ where $u(x, y, z) = xy^2z^3$. **Solution:**

$$\frac{\partial^3}{\partial x \partial y \partial z} u = \frac{\partial^2}{\partial x \partial y}(3xy^2z^2) = \frac{\partial}{\partial x}(6xyz^2) = \boxed{6yz^2}$$

9(a). Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ c & 0 & 3 \end{bmatrix}.$$

(Your answer should be an expression involving c.)

Solution; Expand using column 1:

$$\det A = 1\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + c\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 6 + c(1 \cdot 1 - 2 \cdot 1) = \boxed{6 - c}.$$

9(b). Let U be the ball of radius 1 centered at the origin. For which values of c will U and $A(U)$ have the same volume?

Solution: $\mathrm{vol}(A(U)) = |\det A|\,\mathrm{vol}(U) = |6-c|\,\mathrm{vol}(U)$, so the volumes will be equal precisely when $|6 - c| = 1$, i.e., when $\boxed{c = 5}$ or $\boxed{c = 7}$.

10. Suppose a particle moves with constant speed 5. Prove that at each time t, the particle's velocity and acceleration vectors are perpendicular to each other.

Solution: Let $\mathbf{v}(t)$ denote the velocity. We are given that $\|\mathbf{v}(t)\| = 5$, or, equivalently that $\|\mathbf{v}(t)\|^2 = 25$. Differentiate both sides:

$$\begin{aligned} 0 &= \left(\frac{d}{dt}\right)(\|\mathbf{v}(t)\|^2) \\ &= \left(\frac{d}{dt}\right)(\mathbf{v}(t)\cdot\mathbf{v}(t)) \\ &= 2\mathbf{v}(t)\cdot\mathbf{v}'(t). \end{aligned}$$

Thus the velocity $\mathbf{v}(t)$ and the acceleration $\mathbf{v}'(t)$ are perpendicular.

NOTE: Speed constant does *NOT* imply that velocity is constant! For example, if

$$\mathbf{x}(t) = (\cos t, \sin t)$$

then the velocity $\mathbf{v}(t) = (-\sin t, \cos t)$ is not constant but the speed

$$\|\mathbf{v}(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

is constant.

If you drive on a curved road, your velocity will be changing even if you keep your speed constant.