SOLUTIONS TO MATH 51 MIDTERM 1

January 29, 2004

1. Find all solutions of the following system:

$$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & x_3 & + & 2x_4 & = & 3 \\ & & x_2 & + & x_3 & + & x_4 & = & 3 \\ x_1 & + & x_2 & + & 3x_3 & + & 4x_4 & = & 9 \end{array}$$

Solution. Write the augmented matrix and then use Gaussian elimination:

$$\begin{bmatrix} 1 & -1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 3 \\ 1 & 1 & 3 & 4 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 2 & 2 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$$\begin{array}{rcrcrcrcl} x_1 & & & + & 2x_3 & + & 3x_4 & = & 6 \\ & & x_2 & + & x_3 & + & x_4 & = & 3 \end{array}$$

The free variables are x_3 and x_4, so the solutions are:

$$\boxed{x_1 = 6 - 2x_3 - 3x_4, \qquad x_2 = 3 - x_3 - x_4, \qquad (x_3 \in \mathbf{R},\ x_4 \in \mathbf{R})}$$

2. Let L be the intersection of the two planes

$$x + y + z = 4 \qquad \text{and} \qquad 2x + 3y + z = 9.$$

Find a parametric equation for L.

Solution. Write the augmented matrix and use Gaussian elimination:

$$\begin{vmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 9 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 1 \end{vmatrix}.$$

so $x + 2z = 3$ and $y - z = 1$, or (moving the free variable z to the right hand side) $x = 3 - 2z$ and $y = 1 + z$. Thus the intersection is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 - 2z \\ 1 + z \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

3(a). Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are points in $\mathbf{R}^n$ such that $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 1$ and such that $\mathbf{w} = -\mathbf{u}$. Suppose also that $\mathbf{v}$ is not equal to $\mathbf{u}$ or to $\mathbf{w}$. Prove that the triangle $\Delta\mathbf{uvw}$ has a right angle at $\mathbf{v}$.

Solution. The vector from $\mathbf{u}$ to $\mathbf{v}$ is $\mathbf{v}-\mathbf{u}$. The vector from $\mathbf{w}$ to $\mathbf{v}$ is $\mathbf{v}-\mathbf{w}$. We want to show that these two vectors are orthogonal, so we calculate the dot product:

$$(\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{w}) = (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}+\mathbf{u}) = \mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{u} = \|\mathbf{v}\|^2-\|\mathbf{u}\|^2 = 1^2-1^2 = 0$$

3(b). Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors in $\mathbf{R}^n$ whose norms are 1, 2, and 3, respectively. Suppose each vector is orthogonal (i.e., perpendicular) to each of the other two. Find a scalar c such that the vector

$$\mathbf{x}+c\mathbf{y}-\mathbf{z}$$

is orthogonal to the vector $\mathbf{x}+\mathbf{y}+\mathbf{z}$.

Solution The vectors $\mathbf{x}+\mathbf{y}+\mathbf{z}$ and $\mathbf{x}+c\mathbf{y}-\mathbf{z}$ vectors will be orthogonal provided their dot product is 0. When we multiply it out (i.e., use the distributive property), all the "mixed" terms ($\mathbf{x}\cdot\mathbf{y}$, $\mathbf{x}\cdot\mathbf{z}$, etc.) are 0 by orthogonality. So

$$0 = (\mathbf{x}+\mathbf{y}+\mathbf{z})\cdot(\mathbf{x}+c\mathbf{y}-\mathbf{z}) = \mathbf{x}\cdot\mathbf{x}+c\mathbf{y}\cdot\mathbf{y}-\mathbf{z}\cdot\mathbf{z} = \|\mathbf{x}\|^2+c\|\mathbf{y}\|^2-\|\mathbf{z}\|^2 = 1+4c-9 = 4c-8.$$

Thus $0 = 4c-8$, which means $c = 2$.

4. Consider the points $A = (1,1,1,1)$, $B = (1,2,0,-1)$ and $C = (1,0,-1,1)$ in $\mathbf{R}^4$.

4(a) Find the cosine of the angle at B of the triangle ABC.

Solution: $\overrightarrow{BA} = A-B = (1,1,1,1)-(1,2,0,-1) = (0,-1,1,2)$ and $\overrightarrow{BC} = C-B = (0,-2,-1,2)$, so

$$\begin{aligned}
\|\overrightarrow{BA}\| &= \sqrt{0^2+(-1)^2+1^2+2^2} = \sqrt{6} \\
\|\overrightarrow{BC}\| &= \sqrt{0^2+(-2)^2+(-1)^2+2^2} = \sqrt{4+1+4} = \sqrt{9} = 3 \\
\overrightarrow{BA}\cdot\overrightarrow{BC} &= 0+2-1+4 = 5 \\
\cos\theta &= \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\|\overrightarrow{BA}\|\,\|\overrightarrow{BC}\|} \\
&= \frac{5}{3\sqrt{6}} = \frac{5\sqrt{6}}{18}
\end{aligned}$$

4(b) Find a parametric equation for the plane through the points A, B, and C from part (a).

Answer: $B+s(A-B)+t(C-B)$, or

$$(1,2,0,-1)+s(0,-1,1,2)+t(0,-2,-1,2)$$

5. Are the following three vectors in $\mathbf{R}^3$ linearly independent or linearly dependent? Show your work and explain your answer.

$$\mathbf{u} = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix}-2\\1\\-1\end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix}-1\\8\\7\end{bmatrix}$$

Solution: Solving $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = 0$ by Gaussian elimination gives $x = -3z$ and $y = -2z$, with z free. Thus for example we can let $z = 1$, which gives

$$-3\mathbf{u} - 2\mathbf{v} + \mathbf{w} = 0$$

so the vectors are dependent.

6. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & -1 \end{bmatrix}.$$

What condition(s) must $\mathbf{b}$ satisfy to be in the column space of A?

(Your answer should be one or more equations of the form $?\,b_1 + ?\,b_2 + ?\,b_3 + ?\,b_4 = ?$.)

Solution: We solve $A\mathbf{x} = \mathbf{b}$ by Gaussian elimination, using augmented matrices:

$$\begin{bmatrix} 1 & 2 & 1 & b_1 \\ 2 & 4 & 2 & b_2 \\ 1 & 3 & 3 & b_3 \\ 1 & 1 & -1 & b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 2 & b_3 - b_1 \\ 0 & -1 & -2 & b_4 - b_1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 2 & b_3 - b_1 \\ 0 & 0 & 0 & (b_4 - b_1) + (b_3 - b_1) \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 1 & 2 & b_3 - b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_4 + b_3 - 2b_1 \end{bmatrix}$$

(This last matrix is in row echelon form, not reduced row echelon form. To get rref, we should subtract 2 times row 2 from row 1, but that won't affect the conditions on $\mathbf{b}$.) The conditions that $\mathbf{b}$ must satisfy to be in the column space are given by rows 2 and 3:

$$\boxed{\begin{aligned} b_2 - 2b_1 &= 0 \\ b_4 + b_3 - 2b_1 &= 0 \end{aligned}}$$

7(a) Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent vectors in $\mathbf{R}^n$. Show that if A is an $m \times n$ matrix, then the vectors $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_k$ must also be linearly dependent.

Solution. Since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent, there are coefficients $c_1, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_k\mathbf{v}_k = 0.$$

Switching the right and left sides of the equation, and then multiplying both sides by A gives:

$$\begin{aligned} A0 &= A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_k\mathbf{v}_k) \\ &= A(c_1\mathbf{v}_1) + A(c_2\mathbf{v}_2) + \cdots + A(c_k\mathbf{v}_k) \\ &= c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \cdots + c_k(A\mathbf{v}_k) \end{aligned}$$

Since $A0 = 0$, this means

$$0 = c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \cdots + c_k(A\mathbf{v}_k).$$

Not all of the c_is are 0. Thus $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_k$ are linearly dependent.

Another solution. Since the $\mathbf{v}_i$s are dependent, one, say $\mathbf{v}_k$, is a linear combination of the others:

$$\mathbf{v}_k = d_1\mathbf{v}_1 + \ldots d_{k-1}\mathbf{v}_{k-1}.$$

Multiply both sides by A:

$$\begin{aligned} A\mathbf{v}_k &= A(d_1\mathbf{v}_1 + \cdots + d_{k-1}\mathbf{v}_{k-1}) \\ &= d_1(A\mathbf{v}_1) + \ldots d_{k-1}(A\mathbf{v}_{k-1}). \end{aligned}$$

Thus $A\mathbf{v}_k$ is a linear combination of $A\mathbf{v}_1, \ldots, A\mathbf{v}_{k-1}$. Thus the vectors $A\mathbf{v}_1, \ldots, A\mathbf{v}_k$ are linearly dependent.

7(b) Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly independent vectors in $\mathbf{R}^n$. Prove that $\mathbf{x}$, $\mathbf{x} + \mathbf{y}$, and $\mathbf{x} + \mathbf{y} + \mathbf{z}$ are also linearly independent.

Solution Suppose a linear combination of the vectors in question adds up to 0:

$$(*) \qquad a\mathbf{x} + b(\mathbf{x} + \mathbf{y}) + c(\mathbf{x} + \mathbf{y} + \mathbf{z}) = 0.$$

We must show that this can happen only when $a = b = c = 0$. Multiplying (*) out (using the distributive property) and combining like terms, we get

$$(a + b + c)\mathbf{x} + (b + c)\mathbf{y} + c\mathbf{z} = 0.$$

Since $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are independent, the three coefficients must be 0:

$$a + b + c = b + c = c = 0.$$

Since $c = 0$ and $b + c = 0$, b must also be 0. Since $a + b + c = 0$ and $b = c = 0$, a must also be 0.

8. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

The reduced echelon form for A is

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(You do not need to check this.)

8(a) Find a basis for the column space $C(A)$ of A.

Solution. The pivots in rref(A) are in columns 1, 3, and 4, so the first, third, and fourth columns of A form a basis for $C(A)$:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

8(b) Find a basis for the nullspace $N(A)$ of A.

Solution. The nullspace of A contains all solutions $\mathbf{x}$ of $A\mathbf{x} = 0$. This system of equations is equivalent to $R\mathbf{x} = 0$. From R we see that the free variables are x_2 and x_5, and that $x_1 = -x_2 - 2x_5$, $x_3 = -x_5$, and $x_4 = x_5$. Thus $\mathbf{x}$ is in the nullspace if and only if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 - 2x_5 \\ x_2 \\ -x_5 \\ x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} x_5.$$

Thus $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ are a basis for $N(A)$.

8(c) If $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, then $A\mathbf{v} = \begin{bmatrix} 7 \\ 10 \\ 2 \\ 8 \end{bmatrix}$. (You do not need to check this.) Find all solutions $\mathbf{x}$ of

$$A\mathbf{x} = \begin{bmatrix} 7 \\ 10 \\ 2 \\ 8 \end{bmatrix}.$$

Solution: using part (b) (see proposition 8.2 on p. 52 of the text),

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

9. Let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Compute each the following:

(a) $3\mathbf{x} - 5\mathbf{y}$

Solution. $3\begin{bmatrix}1\\1\\2\end{bmatrix} - 5\begin{bmatrix}2\\2\\-3\end{bmatrix} = \begin{bmatrix}3\\3\\6\end{bmatrix} - \begin{bmatrix}10\\10\\-15\end{bmatrix} = \begin{bmatrix}-7\\-7\\21\end{bmatrix}.$

(b) $\mathbf{x} \cdot (\mathbf{y} + \mathbf{x})$

Solution. $\begin{bmatrix}1\\1\\2\end{bmatrix} \cdot \begin{bmatrix}3\\3\\-1\end{bmatrix} = 3 + 3 - 2 = 4.$

(c) $\|\mathbf{x} - \mathbf{y}\|^2$

Solution. $\mathbf{x} - \mathbf{y} = \begin{bmatrix}-1\\-1\\5\end{bmatrix}$, so the square of the length of this vector is $(-1)^2 + (-1)^2 + 5^2 = 1 + 1 + 25 = 27$.

(d) $A\mathbf{y}$. Solution:

$$\begin{bmatrix}0 & 2 & -2\\1 & 1 & 4\end{bmatrix}\begin{bmatrix}2\\2\\-3\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} + \begin{bmatrix}4\\2\end{bmatrix} + \begin{bmatrix}6\\-12\end{bmatrix} = \begin{bmatrix}10\\-8\end{bmatrix}.$$

(e) $B(A\mathbf{x})$. Solution:

$$\begin{bmatrix}2 & 3\\1 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}0 & 2 & -2\\1 & 1 & 4\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}2 & 3\\1 & 1\\-1 & 2\end{bmatrix}\left(\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}-4\\8\end{bmatrix}\right)$$
$$= \begin{bmatrix}2 & 3\\1 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}-2\\10\end{bmatrix} = \begin{bmatrix}-4\\-2\\2\end{bmatrix} + \begin{bmatrix}30\\10\\20\end{bmatrix} = \begin{bmatrix}26\\8\\22\end{bmatrix}.$$

10(a,b,c). Suppose V is a set of vectors in $\mathbf{R}^n$. What three properties must V have in order to be a linear subspace of $\mathbf{R}^n$?

Solution. (1) V must contain the 0 vector, (2) V must be closed under addition, and (3) V must be closed under scalar multiplication.

10(d,e). State whether each of the following sets is a linear subspace of $\mathbf{R}^2$. If it is not, explain why not.

(d). The set W of vectors $\begin{bmatrix}x\\y\end{bmatrix}$ such that $x \geq 0$.

Solution. Not a subspace because it is not closed under scalar multiplication. For example, $\begin{bmatrix}1\\2\end{bmatrix}$ is in W, but $(-1)\begin{bmatrix}1\\2\end{bmatrix}$ is not in W.

(e). The set U of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that x is a whole number.

Solution. Not a subspace because it is not closed under scalar multiplication. For example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in W, but $(1/2)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in W.