MATH 51 MIDTERM 1

January 29, 2004

1. Find all solutions of the following system:

$$\begin{array}{ccccccccc} x_1 & - & x_2 & + & x_3 & + & 2x_4 & = & 3 \\ & & x_2 & + & x_3 & + & x_4 & = & 3 \\ x_1 & + & x_2 & + & 3x_3 & + & 4x_4 & = & 9 \end{array}$$

2. Let L be the intersection of the two planes

$$x + y + z = 4 \qquad \text{and} \qquad 2x + 3y + z = 9.$$

Find a parametric equation for L.

3(a) Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are points in $\mathbf{R}^n$ such that $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 1$ and such that $\mathbf{w} = -\mathbf{u}$. Suppose also that $\mathbf{v}$ is not equal to $\mathbf{u}$ or to $\mathbf{w}$. Prove that the triangle $\Delta\mathbf{uvw}$ has a right angle at $\mathbf{v}$.

3(b) Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors in $\mathbf{R}^n$ whose norms are 1, 2, and 3, respectively. Suppose each vector is orthogonal (i.e., perpendicular) to each of the other two. Find a scalar c such that the vector

$$\mathbf{x} + c\mathbf{y} - \mathbf{z}$$

is orthogonal to the vector $\mathbf{x} + \mathbf{y} + \mathbf{z}$.

4. Consider the points $A = (1, 1, 1, 1)$, $B = (1, 2, 0, -1)$ and $C = (1, 0, -1, 1)$ in $\mathbf{R}^4$.

4(a) Find the cosine of the angle at B of the triangle ABC.

4(b) Find a parametric equation for the plane through the points A, B, and C from part (a).

5. Are the following three vectors in $\mathbf{R}^3$ linearly independent or linearly dependent? Show your work and explain your answer.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} -1 \\ 8 \\ 7 \end{bmatrix}$$

6. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & -1 \end{bmatrix}.$$

What condition(s) must $\mathbf{b}$ satisfy to be in the column space of A?

(Your answer should be one or more equations of the form $?\,b_1 + ?\,b_2 + ?\,b_3 + ?\,b_4 = ?$.)

7(a) Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent vectors in $\mathbf{R}^n$. Show that if A is an $m \times n$ matrix, then the vectors $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_k$ must also be linearly dependent.

7(b) Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly independent vectors in $\mathbf{R}^n$. Prove that $\mathbf{x}$, $\mathbf{x} + \mathbf{y}$, and $\mathbf{x} + \mathbf{y} + \mathbf{z}$ are also linearly independent.

8. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

The reduced echelon form for A is

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(You do not need to check this.)

8(a) Find a basis for the column space $C(A)$ of A.

8(b) Find a basis for the nullspace $N(A)$ of A.

8(c) If $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, then $A\mathbf{v} = \begin{bmatrix} 7 \\ 10 \\ 2 \\ 8 \end{bmatrix}$. (You do not need to check this.) Find all solutions $\mathbf{x}$ of

$$A\mathbf{x} = \begin{bmatrix} 7 \\ 10 \\ 2 \\ 8 \end{bmatrix}.$$

9. Let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Compute each the following:

(a) $3\mathbf{x} - 5\mathbf{y}$

(b) $\mathbf{x} \cdot (\mathbf{y} + \mathbf{x})$

(c) $\|\mathbf{x} - \mathbf{y}\|^2$

(d) $A\mathbf{y}$

(e) $B(A\mathbf{x})$

10(a,b,c). Suppose V is a set of vectors in $\mathbf{R}^n$. What three properties must V have in order to be a linear subspace of V?

10(d,e). State whether each of the following sets is a linear subspace of $\mathbf{R}^2$. If it is not, explain why not.

(d). The set W of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x \geq 0$.

(e). The set U of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that x is an integer.