EXAM 2

Math 51, Spring 2004.

You have 2 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name Solutions

ID number____________________

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________

1. __________ (/30 points)

2. __________ (/30 points)

3. __________ (/30 points)

4. __________ (/30 points)

5. __________ (/30 points)

Bonus __________ (/15 points)

Total __________ (/150 points)

Circle your TA's name:

Brett Parker (2 and 6)

Chad Groft (3 and 7)

Joe Blitzstein (4 and 8)

Ryan Vinroot (ACE)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Find the determinant and the inverse of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 7 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 7 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -7 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{l} r_1 \\ r_2 \\ r_3 \\ r_4 + 2r_2 - 7r_1 \\ r_5 \end{array}$$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 14/5 & 1/5 & 0 & -2/5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & -7/5 & 2/5 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} r_1 \\ r_2 - \frac{2}{5}r_4 \\ r_3 - 3r_5 \\ r_4/5 \\ r_5 \end{array}$$

this is the inverse matrix.

The only row operation performed that affects determinant was dividing the 4th row by 5. So, $(\det A)/5 = \det(I_5) = 1$

Thus $\boxed{\det A = 5}$

2. Use the fact that

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

to compute

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{10}$$

First, observe that $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{10} = \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)^{10}$

$$= \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\cancel{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \right) \cancel{\cdots} \left(\cancel{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\right)^{10}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1024 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1024 \\ 0 & 1024 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} 1 & -1023 \\ 0 & 1024 \end{pmatrix}}$$

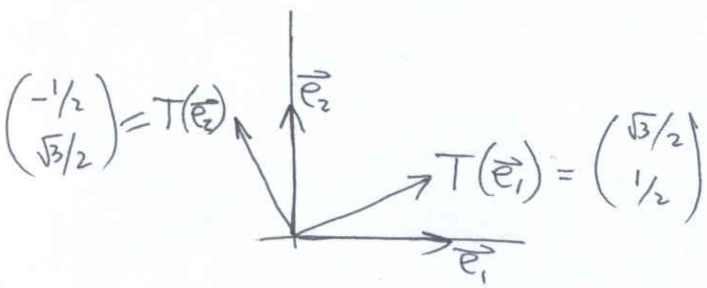
3. Find the equation of the ellipse obtained by rotating (clockwise around the origin by angle $\pi/6$) the ellipse with equation

$$\frac{x^2}{4} + y^2 = 1$$

Hint: A point $\begin{bmatrix} x \\ y \end{bmatrix}$ *on this new ellipse is defined by the property that, if rotated* **counter-clockwise** *around the origin, the resulting point* $\begin{bmatrix} u \\ v \end{bmatrix}$ *will be on the original ellipse and thus will satisfy the equation*

$$\frac{u^2}{4} + v^2 = 1$$

Let T rotate counterclockwise around $\vec{0}$ by $\pi/6$:



$$\Longrightarrow A = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

is the matrix with $T(\vec{x}) = A\vec{x}$

As stated in the hint, $\begin{pmatrix} x \\ y \end{pmatrix}$ is on the ellipse in question

iff $\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix}$ satisfies $\frac{u^2}{4} + v^2 = 1$

And $\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y \end{pmatrix}$

So the equation is

$$\frac{\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)^2}{4} + \left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^2 = 1$$

which will simplify as

(over)

$$\frac{\left(\frac{3}{4}x^2 + \frac{1}{4}y^2 - \frac{\sqrt{3}}{2}xy\right)}{4} + \left(\frac{1}{4}x^2 + \frac{3}{4}y^2 + \frac{\sqrt{3}}{2}xy\right) = 1$$

$$\boxed{\frac{7}{16}x^2 + \frac{13}{16}y^2 + \frac{3\sqrt{3}}{8}xy = 1}$$

4. Let A be an $m \times n$ matrix. Describe the precise procedure by which you would determine bases for the column space and the null space of A. Based on this description, prove the Rank-Nullity Theorem, which states that

$$\dim(C(A)) + \dim(N(A)) = n$$

Make sure to explain clearly all of your reasoning.

To get a basis for $C(A)$, we take the columns of A that correspond to the columns of $\text{rref}(A)$ that contain pivots.

To get a basis for $N(A)$, we solve for pivot variables in terms of free variables in the system $\text{rref}(A)\vec{x} = \vec{0}$; the solution set can then be written explicitly in terms of the free variables, and by separating and factoring, we can write the solution set as the span of a set of vectors — these vectors are a basis for $N(A)$.

From the above we see that for $C(A)$ we get a basis vector for each pivot column in $\text{rref}(A)$; and for $N(A)$ we get a basis vector for each free column in $\text{rref}(A)$.

Thus the total number of elements in the two bases

(over)

together is the total number of pivot and free columns, which is the total number of columns.

Since the number of elements in a basis is dimension, we get

$$\left(\#\text{ vects in basis for } C(A)\right) + \left(\#\text{vects in basis for } N(A)\right) = \#\text{ cols.}$$

$$\Rightarrow \quad \boxed{\dim\left(C(A)\right) + \dim\left(N(A)\right) = n}$$

5. Prove that for every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, there exists a matrix A such that for all $\overrightarrow{x} \in \mathbb{R}^n$,

$$T(\overrightarrow{x}) = A\overrightarrow{x}$$

Given the linear transformation T, define a matrix A by

$$A = \begin{pmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{pmatrix}$$

In other words, let the columns of A be defined as the images of the standard basis vectors $\vec{e}_1, \ldots, \vec{e}_n$ by T.

Then: ① $T(\vec{x}) = T(x_1\vec{e}_1 + \ldots + x_n\vec{e}_n)$

$$= x_1 T(\vec{e}_1) + \ldots + x_n T(\vec{e}_n)$$

(using the linearity of T)

② $A\vec{x} = x_1 (\text{1st col. of } A) + \ldots + x_n (\text{nth col. of } A)$

(property of matrix-vect prods)

$$= x_1 T(\vec{e}_1) + \ldots + x_n T(\vec{e}_n)$$

These expressions are identical — so, for all $\vec{x} \in \mathbb{R}^n$,

$$T(\vec{x}) = A\vec{x}$$

Bonus Question: Find a matrix A such that

(1) None of $A, A^2, A^3, \ldots, A^9$ is the zero matrix
(2) A^{10} is the zero matrix.

Let T be the linear transformation defined by

$$\begin{aligned} \vec{e}_1 &\longmapsto \vec{e}_2 \\ \vec{e}_2 &\longmapsto \vec{e}_3 \\ &\vdots \\ \vec{e}_9 &\longmapsto \vec{e}_{10} \\ \vec{e}_{10} &\longmapsto \vec{0} \end{aligned}$$

In other words, $T: \mathbb{R}^{10} \to \mathbb{R}^{10}$ sends standard basis vectors to standard basis vectors, "moving them over" by one position, and the one at the end goes to $\vec{0}$.

$$\vec{e}_1 \curvearrowright \vec{e}_2 \curvearrowright \vec{e}_3 \curvearrowright \vec{e}_4 \curvearrowright \cdots \curvearrowright \vec{e}_9 \curvearrowright \vec{e}_{10} \curvearrowright \vec{0}$$

The matrix for this transformation is below, with ones just below the diagonal, and zeroes everywhere else.

$$A = \begin{pmatrix} 0 & 0 & \vdots & & & \\ 1 & 0 & \vdots & & & \\ 0 & 1 & 0 & \ddots & & \\ \vdots & 0 & 1 & \ddots & & \\ \vdots & \vdots & 0 & & \ddots & \\ \vdots & \vdots & \vdots & & & \\ \vdots & \vdots & \vdots & & 1 & 0 \end{pmatrix}$$

Raising this matrix to a power corresponds to composing the transformation with itself, which results simply in "moving them over" by more than one position:

$$A^2 = T \circ T: \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \quad \vec{e}_4 \quad \vec{e}_5 \quad \cdots \quad \vec{e}_9 \quad \vec{e}_{10} \quad \vec{0}$$

$$A^3 = T \circ T \circ T: \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \quad \vec{e}_4 \quad \vec{e}_5 \quad \cdots \quad \vec{e}_9 \quad \vec{e}_{10} \quad \vec{0}$$

$$\vdots$$

$$A^9 = T \circ \cdots \circ T: \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \quad \cdots \quad \vec{e}_9 \quad \vec{e}_{10} \quad \vec{0}$$

$$A^{10} = T \circ \cdots \circ T: \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \quad \cdots \quad \vec{e}_9 \quad \vec{e}_{10} \quad \vec{0}$$

The transformations corresponding to $A, A^2, \ldots, A^9$ send $\vec{e}_1$ to a nonzero vector, so those vectors can't be the zero matrix. But A^{10} sends everything to $\vec{0}$ — so it is the zero matrix.

(Algebraically, as we raise A to higher powers, the line of 1's goes lower and lower; A^9 has a 1 only in the bottom left position. Finally A^{10} is just all zeroes.)