

EXAM II SOLUTIONS

Math 51, Spring 2003.

You have 2 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name _____

ID number _____

1. _____ (/30 points)

2. _____ (/30 points)

3. _____ (/30 points)

4. _____ (/30 points)

5. _____ (/30 points)

Bonus _____ (/15 points)

Total _____ (/150 points)

“On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

Signature: _____

Circle your TA's name:

Byoung-du Kim (2 and 6)

Ted Hwa (3 and 7)

Jacob Shapiro (4 and 8)

Ryan Vinroot (A02)

Michel Grueneberg (A03)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. (a) Find bases for the kernel and the image of the linear transformation given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 3x - y + z \\ y + 2z \\ 3y + 6z \end{bmatrix}$$

Solution: Of course the kernel of the linear transformation is just the null space of the corresponding matrix; and similarly, the image is the column space. So we will solve this problem in terms of matrices.

The matrix in question has columns which are the images of the standard basis vectors:

$$T(e_1) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \quad T(e_2) = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \quad T(e_3) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

So the matrix is

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{pmatrix}$$

This we can row reduce to its reduced row echelon form

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The solutions to the homogeneous system are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -2z \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

and so a basis for the null space (and thus the kernel of the linear transformation) is

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

And since there are pivots in the first and second columns of $\text{rref}(A)$, a basis for the column space (and thus the image of the linear transformation) is

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

- (b) Find any nonzero vector \vec{x} with the property that \vec{x} is perpendicular to every vector in the kernel of T ; and explain how you know that the vector you supply has this property.

Solution: Vectors in the row space are always perpendicular to vectors in the null space, which of course is equal to the kernel of the linear transformation. So, any vector in the row space will suffice for this problem. For example, any row vector will do:

$$\vec{r}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

2. (a) Suppose that A is a 4×3 matrix, and that $C(A)$ has dimension 2. What is the dimension of $N(A)$?

Solution: The matrix A is a function from \mathbb{R}^3 to \mathbb{R}^4 . So, the Rank-Nullity theorem tells us that

$$\dim C(A) + \dim N(A) = 3$$

Since the dimension of $C(A)$ is given to be 2, we know that $N(A)$ must have dimension 1.

- (b) Suppose that $N(M_2) = \{\vec{0}\}$. Show that $N(M_2M_1) = N(M_1)$.

Solution: We need to show that

$$M_2M_1\vec{x} = \vec{0} \iff M_1\vec{x} = \vec{0}$$

(\Leftarrow) If $M_1\vec{x} = \vec{0}$, then $M_2M_1\vec{x} = M_2(M_1\vec{x}) = M_2\vec{0} = \vec{0}$.

(\Rightarrow) Now suppose that $M_2M_1\vec{x} = \vec{0}$. Of course this means that $M_2(M_1\vec{x}) = \vec{0}$. Since we are given that the null space of M_2 is just the zero vector, this means that $(M_1\vec{x}) = \vec{0}$, as desired.

- (c) Suppose that M_2 is a 3×2 matrix, M_1 is a 2×4 matrix, the rank of M_2 is 2, and the rank of M_2M_1 is 1. What is the dimension of $C(M_1)$?

Suggestions: Use part (b) and the Rank-Nullity Theorem to determine the dimensions of $N(M_2)$ and $N(M_2M_1)$, and then deduce the dimension of $N(M_1)$.

Solution: We have that $M_1 : \mathbb{R}^4 \rightarrow \mathbb{R}^2$, and $M_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$. So, we conclude that $M_2M_1 : \mathbb{R}^4 \rightarrow \mathbb{R}^3$. We then can use the Rank-Nullity Theorem as suggested to conclude

$$\text{rank}M_2 = 2 \implies \dim C(M_2) = 2 \implies \dim N(M_2) = 0 \implies N(M_2) = \{\vec{0}\}$$

$$\text{rank}M_2M_1 = 1 \implies \dim C(M_2M_1) = 1 \implies \dim N(M_2M_1) = 3$$

Since $N(M_2) = \{\vec{0}\}$, part (b) then tells us that $N(M_1) = N(M_2M_1)$, and so we conclude that $\dim N(M_1) = 3$.

Applying the Rank-Nullity Theorem to M_1 , we conclude that $\dim C(M_1) = 1$.

3. (a) Compute the determinant of the following matrix:

$$A = \begin{pmatrix} 3 & -4 & 7 & 9 & 5 \\ 4 & 0 & 0 & 0 & 0 \\ 17 & 21 & -5 & 11 & 6 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix}$$

(Suggestion: Use the properties of the determinant to simplify this question before you begin with the computations.)

Solution: As suggested, we first switch the rows around... in particular so that the rows with mostly zeroes are on top, and the nonzero components of those rows are along the diagonal. This requires three row exchanges, so the determinant is changed by a factor of $(-1)^3 = -1$. We get

$$\det \begin{pmatrix} 3 & -4 & 7 & 9 & 5 \\ 4 & 0 & 0 & 0 & 0 \\ 17 & 21 & -5 & 11 & 6 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix} = -\det \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 17 & 21 & -5 & 11 & 6 \\ 3 & -4 & 7 & 9 & 5 \end{pmatrix}$$

At this point, we can either compute directly by minors, or row reduce.

Using minors, we get

$$\begin{aligned} &= - \left(4 \det \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 21 & -5 & 11 & 6 \\ -4 & 7 & 9 & 5 \end{pmatrix} - 0 + 0 - 0 + 0 \right) \\ &= - \left(4 \left(3 \det \begin{pmatrix} 2 & 0 & 0 \\ -5 & 11 & 6 \\ 7 & 9 & 5 \end{pmatrix} - 0 + 0 - 0 \right) - 0 + 0 - 0 + 0 \right) \\ &= - \left(4 \left(3 \left(2 \det \begin{pmatrix} 11 & 6 \\ 9 & 5 \end{pmatrix} - 0 + 0 \right) - 0 + 0 - 0 \right) - 0 + 0 - 0 + 0 \right) \\ &= -(4(3(2(1)))) \\ &= -24 \end{aligned}$$

If we were to row reduce instead, the first thing we would do is add appropriate multiples of the top three rows to the bottom two rows to eliminate the first three components of the bottom two rows; since adding multiples of a row to another row does not change the determinant, we get simply

$$= -\det \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 17 & 21 & -5 & 11 & 6 \\ 3 & -4 & 7 & 9 & 5 \end{pmatrix} = -\det \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 11 & 6 \\ 0 & 0 & 0 & 9 & 5 \end{pmatrix}$$

Using the linearity of determinant in each row, we get

$$= -24 \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 11 & 6 \\ 0 & 0 & 0 & 9 & 5 \end{pmatrix} = -\frac{24}{11} \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 11 & 6 \\ 0 & 0 & 0 & 99 & 55 \end{pmatrix}$$

Again adding multiples of rows to other rows, this becomes

$$= -\frac{24}{11} \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 11 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = -\frac{24}{11} \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= -24 \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = -24$$

(b) Is the above matrix A invertible? Why or why not?

Solution: The determinant of the matrix is non-zero, so the matrix IS invertible.

(c) Suppose B is a 2x2 matrix with determinant 3, and that S is a set in \mathbb{R}^2 such that the area of $B(S)$ is 10.

What is the area of S ?

Solution:

$$\text{area}(S) \cdot |\det B| = \text{area}(B(S))$$

$$\text{area}(S) \cdot |3| = 10$$

$$\text{area}(S) = \frac{10}{3}$$

4. Suppose that B is an invertible 3×3 matrix, but we are only given the first two columns:

$$B = \begin{pmatrix} 1 & 0 & ? \\ 0 & 0 & ? \\ -4 & 2 & ? \end{pmatrix}$$

(a) Is the above enough information to determine the first column of B^{-1} ? If so, find that first column; if not, explain why you cannot find it.

(Hint: What is $B^{-1} \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$? What is $B^{-1} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$?)

Solution: The two given columns of B tell us that

$$B \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \quad B \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

This tells us that

$$B^{-1} \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad B^{-1} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

If we add the first equation to twice the second equation, we get

$$B^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

which is of course the first column of B^{-1} .

- (b) Is the above enough information to determine the second column of B^{-1} ? If so, find that second column; if not, explain why you cannot find it.

Solution: We were able to find the first column of B^{-1} (in other words, $B^{-1}(\vec{e}_1)$) because \vec{e}_1 is a linear combination of the two vectors for which we know the image of B^{-1} ; namely,

$$\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

However, we immediately note that \vec{e}_2 is NOT a linear combination of those vectors, because they are both zero in their second coordinate.

So, we cannot use this method to determine the image of \vec{e}_2 .

In fact, some experimenting shows that $B^{-1}e_2$ is definitely not uniquely determined by the given information. For example:

- i. If

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 2 & 0 \end{pmatrix}$$

then $B(\vec{e}_3) = \vec{e}_2$, and thus the second column of B^{-1} is $B^{-1}\vec{e}_2 = \vec{e}_3$.

- ii. However, if

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -4 & 2 & 0 \end{pmatrix}$$

then $B(\vec{e}_3) = -\vec{e}_2$, and thus the second column of B^{-1} is $B^{-1}\vec{e}_2 = -\vec{e}_3$.

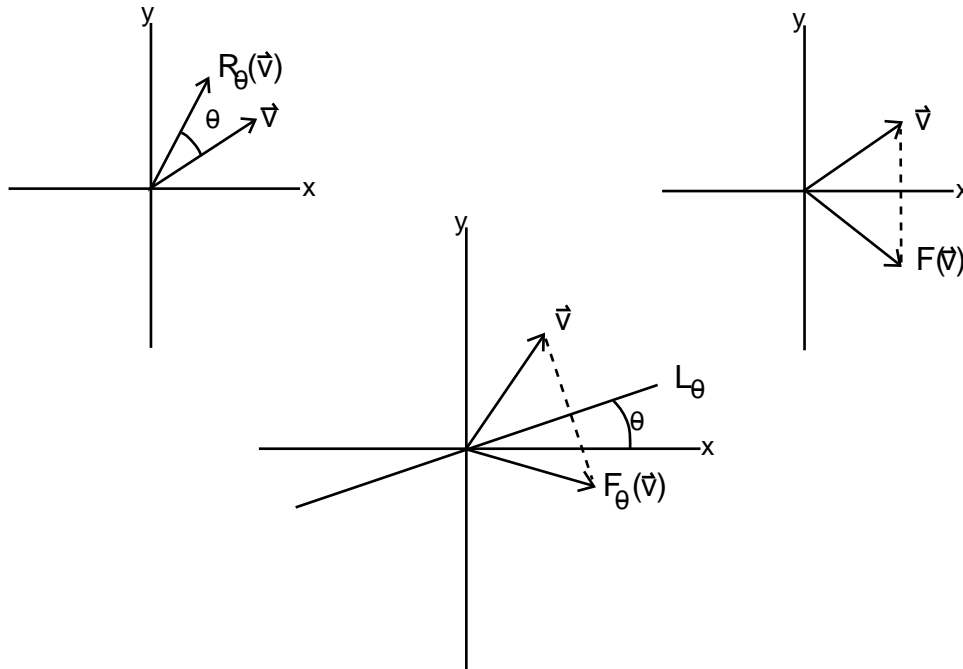
So, we clearly cannot come to any firm conclusions about the second column of B^{-1} .

5. For this problem, we define the following linear transformations in the plane:

R_θ = rotation counter-clockwise around the origin by an angle θ

F_θ = flip (reflection) over the line L_θ obtained by rotating the x -axis counterclockwise by the angle θ

$F = F_0$ = flip over the x -axis itself



(a) Write down the matrices that correspond to R_θ and F .

Solution: The columns of R_θ are the images of the standard basis vectors. If we rotate the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ counterclockwise by the angle θ , we get $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$; and if we rotate the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ counterclockwise by the angle θ , we get $\begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$. So the matrix is

$$A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

For the flip F , we have that the image of \vec{e}_1 is \vec{e}_1 , and the image of \vec{e}_2 is $-\vec{e}_2$. So the matrix is

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) Show that for any angle θ , we have $R_\theta \circ F = F \circ R_{-\theta}$

Solution: To show this statement about compositions of linear transformations, we can simply prove the corresponding statement for matrices:

$$A_\theta B = BA_{-\theta}$$

We can just compute both sides of this equation directly:

$$\begin{aligned} A_\theta B &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA_{-\theta} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \end{aligned}$$

(c) It can be shown also that $F_\theta = R_\theta \circ F \circ R_{-\theta}$. Use this fact (you do not need to prove it) and the result from part (b) to answer the following:

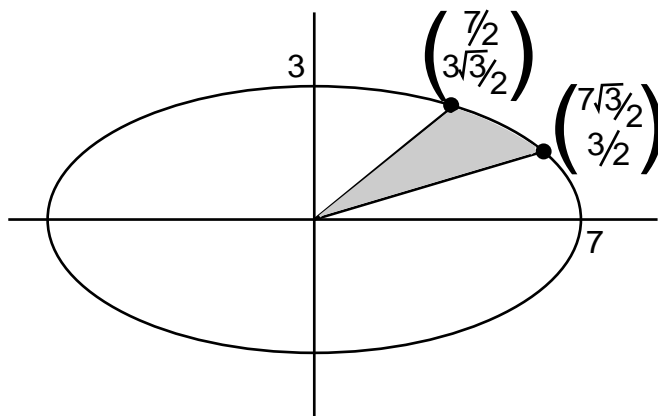
The composition $F_\alpha \circ F_\beta$ of two flips is the same as a rotation around the origin by what angle?

Solution: We use the given equation, note that a composition of two rotations is just a rotation by the sum of the two angles, and then use part (b):

$$\begin{aligned} F_\alpha \circ F_\beta &= (R_\alpha \circ F \circ R_{-\alpha}) \circ (R_\beta \circ F \circ R_{-\beta}) \\ &= R_\alpha \circ F \circ R_{-\alpha} \circ R_\beta \circ F \circ R_{-\beta} \\ &= R_\alpha \circ F \circ R_{\beta-\alpha} \circ F \circ R_{-\beta} \\ &= R_\alpha \circ F \circ (R_{\beta-\alpha} \circ F) \circ R_{-\beta} \\ &= R_\alpha \circ F \circ (F \circ R_{\alpha-\beta}) \circ R_{-\beta} \\ &= R_\alpha \circ R_{\alpha-\beta} \circ R_{-\beta} \\ &= R_{2\alpha-2\beta} \end{aligned}$$

So these two flips are equivalent to a rotation around the origin by an angle $2\alpha - 2\beta$.

Bonus Question: Use linear algebra to help you find the area of the shaded region below, bounded by an ellipse centered at the origin, and two lines (make sure to be explicit in how you are using linear algebra results!).



Solution: An ellipse is just a stretched circle; in particular, the ellipse in question is the image of the unit circle by the linear transformation whose matrix is

$$A = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}$$

And the points indicated on the given ellipse are the images by the same transformation of the points

$$\begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}, \quad \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

which of course correspond to the angles $\pi/6$ and $\pi/3$ on the unit circle.

So, we conclude that the indicated area is in fact just the image under this same linear transformation of the sector S of the unit circle between the angles $\pi/6$ and $\pi/3$; and of course S is just one twelfth of the unit circle.

So, the area of the shaded region $A(S)$ is then

$$\begin{aligned} \text{area}(A(S)) &= |\det A| \cdot \text{area}(S) \\ &= |21| \left(\pi \left(\frac{1}{12} \right) \right) \\ &= \frac{7\pi}{4} \end{aligned}$$