EXAM II

Math 51, Spring 2003.

You have 2 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ______________________________

ID number ____________________

1. __________ (/30 points)

2. __________ (/30 points)

3. __________ (/30 points)

4. __________ (/30 points)

5. __________ (/30 points)

Bonus __________ (/15 points)

Total __________ (/150 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________

Circle your TA's name:

Byoung-du Kim (2 and 6)

Ted Hwa (3 and 7)

Jacob Shapiro (4 and 8)

Ryan Vinroot (A02)

Michel Grueneberg (A03)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. (a) Find bases for the kernel and the image of the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x - y + z \\ y + 2z \\ 3y + 6z \end{bmatrix}$$

(b) Find any nonzero vector $\overrightarrow{x}$ with the property that $\overrightarrow{x}$ is perpendicular to every vector in the kernel of T; and explain how you know that the vector you supply has this property.

2. (a) Suppose that A is a 4x3 matrix, and that $C(A)$ has dimension 2. What is the dimension of $N(A)$?

(b) Suppose that $N(M_2) = \{\overrightarrow{0}\}$. Show that $N(M_2 M_1) = N(M_1)$.

(c) Suppose that M_2 is a 3x2 matrix, M_1 is a 2x4 matrix, the rank of M_2 is 2, and the rank of $M_2 M_1$ is 1. What is the dimension of $C(M_1)$?

Suggestions: Use part (b) and the Rank-Nullity Theorem to determine the dimensions of $N(M_2)$ *and* $N(M_2 M_1)$*, and then deduce the dimension of* $N(M_1)$*.*

3. (a) Compute the determinant of the following matrix:

$$A = \begin{pmatrix} 3 & -4 & 7 & 9 & 5 \\ 4 & 0 & 0 & 0 & 0 \\ 17 & 21 & -5 & 11 & 6 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix}$$

(Suggestion: Use the properties of the determinant to simplify this question before you begin with the computations.)

(b) Is the above matrix A invertible? Why or why not?

(c) Suppose B is a 2x2 matrix with determinant 3, and that S is a set in $\mathbb{R}^2$ such that the area of $B(S)$ is 10.

What is the area of S?

4. Suppose that B is an invertible 3x3 matrix, but we are only given the first two columns:

$$B = \begin{pmatrix} 1 & 0 & ? \\ 0 & 0 & ? \\ -4 & 2 & ? \end{pmatrix}$$

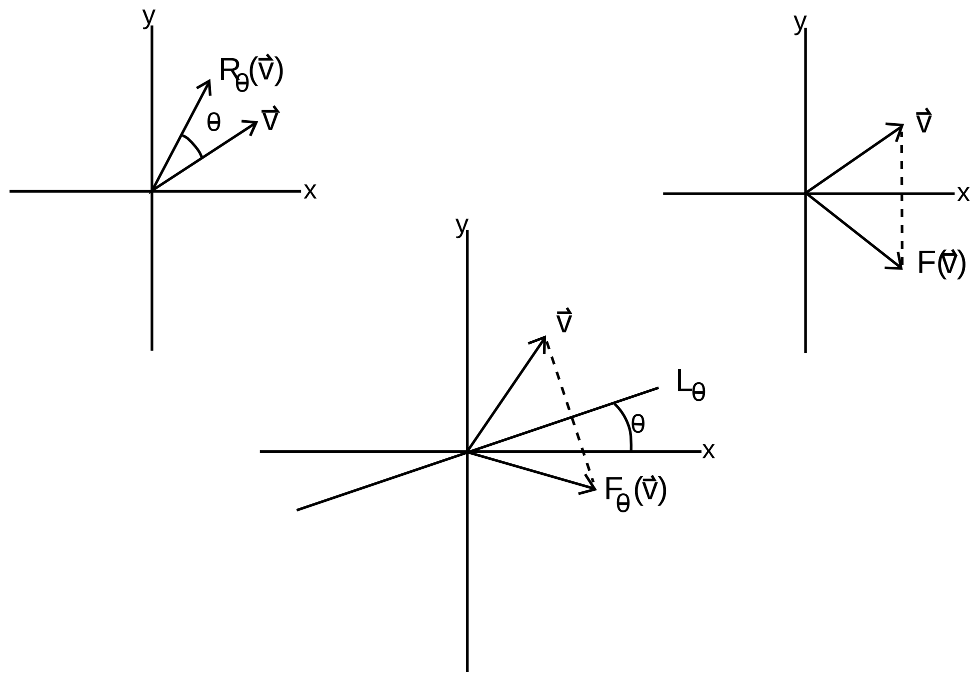
(a) Is the above enough information to determine the first column of B^{-1}? If so, find that first column; if not, explain why you cannot find it.

(Hint: What is $B^{-1}\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$*? What is* $B^{-1}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$*?)*

(b) Is the above enough information to determine the second column of B^{-1}? If so, find that second column; if not, explain why you cannot find it.

5. For this problem, we define the following linear transformations in the plane:

$$\begin{aligned} R_\theta &= \text{rotation counter-clockwise around the origin by an angle } \theta \\ F_\theta &= \text{flip (reflection) over the line } L_\theta \text{ obtained by rotating the } x\text{-axis counterclockwise} \\ &\quad\ \text{by the angle } \theta \\ F &= F_0 = \text{flip over the } x\text{-axis itself} \end{aligned}$$



(a) Write down the matrices that correspond to R_θ and F.

(b) Show that for any angle θ, we have $R_\theta \circ F = F \circ R_{-\theta}$

(c) It can be shown also that $F_\theta = R_\theta \circ F \circ R_{-\theta}$. Use this fact (you do not need to prove it) and the result from part (b) to answer the following:

The composition $F_\alpha \circ F_\beta$ of two flips is the same as a rotation around the origin by what angle?

<u>Bonus Question</u>: Use linear algebra to help you find the area of the shaded region below, bounded by an ellipse centered at the origin, and two lines (make sure to be explicit in how you are using linear algebra results!).