EXAM I SOLUTIONS

Math 51, Spring 2003.

You have 2 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ________________________________

ID number ______________________

1. __________ (/20 points)

2. __________ (/15 points)

3. __________ (/15 points)

4. __________ (/20 points)

5. __________ (/15 points)

6. __________ (/15 points)

Bonus __________ (/10 points)

Total __________ (/100 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________

Circle your TA's name:

Byoung-du Kim (2 and 6)

Ted Hwa (3 and 7)

Jacob Shapiro (4 and 8)

Ryan Vinroot (A02)

Michel Grueneberg (A03)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Consider the following three vectors:

$$\overrightarrow{u} = \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix} \quad \overrightarrow{v} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \quad \overrightarrow{w} = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$$

(a) Find the magnitude of each of the given vectors.

Solution:

$$\begin{aligned} \|\overrightarrow{u}\| &= \sqrt{3^2 + 4^2 + 12^2} = 13 \\ \|\overrightarrow{v}\| &= \sqrt{2^2 + 3^2 + 6^2} = 7 \\ \|\overrightarrow{w}\| &= \sqrt{3^2 + 6^2 + 6^2} = 9 \end{aligned}$$

(b) Find a parametric representation of the unique plane that contains all of the given vectors.

Solution: For the parametric representation we need to have two independent vectors which are parallel to the plane in question, and one point which is in the plane in question. Of course, any of the three given vectors can be used as the point in the plane.

To find the vectors parallel to the plane, we subtract (arbitrarily) $\overrightarrow{v}$ from $\overrightarrow{u}$ and $\overrightarrow{w}$. This gives us

$$\begin{aligned} \overrightarrow{u} - \overrightarrow{v} &= \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} \\ \overrightarrow{w} - \overrightarrow{v} &= \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \end{aligned}$$

Our parametric representation of the plane is then

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}$$

(c) Find an equation representing the unique plane that contains all of the given vectors.

Solution: To find the equation of the given plane, we need to find a normal vector. Since we already have two vectors parallel to the plane, their cross product must be perpendicular to the plane:

$$\overrightarrow{n} = (\overrightarrow{u} - \overrightarrow{v}) \times (\overrightarrow{w} - \overrightarrow{v}) = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -18 \\ 6 \\ 2 \end{bmatrix}$$

The equation of the plane is then

$$\begin{aligned} \overrightarrow{n} \cdot \overrightarrow{x} &= \overrightarrow{n} \cdot \overrightarrow{v} \\ -18x + 6y + 2z &= -6 \end{aligned}$$

or

$$9x - 3y - z = 3$$

2. Find a parametric representation of the complete set of solutions to the following system of equations.

$$\begin{array}{rcrcrcr} -2x & - & y & + & 5z & = & -5 \\ 3x & + & y & - & 7z & = & 6 \\ 2x & & & - & 4z & = & 2 \end{array}$$

Solution: First we write the augmented matrix that corresponds to the given system:

$$\begin{pmatrix} -2 & -1 & 5 & | & -5 \\ 3 & 1 & -7 & | & 6 \\ 2 & 0 & -4 & | & 2 \end{pmatrix}$$

We then use row operations to reduce to the RREF:

$$\begin{pmatrix} 2 & 0 & -4 & | & 2 \\ 3 & 1 & -7 & | & 6 \\ -2 & -1 & 5 & | & -5 \end{pmatrix} \begin{matrix} r_3 \\ r_2 \\ r_1 \end{matrix}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 & | & 1 \\ 3 & 1 & -7 & | & 6 \\ -2 & -1 & 5 & | & -5 \end{pmatrix} \begin{matrix} r_1/2 \\ r_2 \\ r_3 \end{matrix}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 & | & 1 \\ 0 & 1 & -1 & | & 3 \\ 0 & -1 & 1 & | & -3 \end{pmatrix} \begin{matrix} r_1 \\ r_2 - 3r_1 \\ r_3 + 2r_1 \end{matrix}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 & | & 1 \\ 0 & 1 & -1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 + r_2 \end{matrix}$$

We have two pivots, and thus two pivot variables (in this case x and y) and thus one free variable (z). Solving for pivot variables in terms of free variables, we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+2z \\ 3+z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

3. (a) Find two linearly independent vectors that are perpendicular to the vector

$$\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

Solution: We are asked to find two vectors that satisfy the condition $\overrightarrow{x} \cdot \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = 0.$

Said differently, we are asked to find two solutions to the equation

$$-x + 2y + 5z = 0 \quad \Longleftrightarrow \quad x - 2y - 5z = 0$$

We have one pivot variable (x), which we can solve for in terms of the two free variables (y and z):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y + 5z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

So two examples of solutions would be $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$.

(b) Use your answer from part (a) to find a system of equations for which the following parametric line is the *complete* set of solutions.

$$\begin{bmatrix}3\\1\\2\end{bmatrix} + z\begin{bmatrix}-1\\2\\5\end{bmatrix}$$

Solution: This is a line in $\mathbb{R}^3$, so the equations in question must have three variables. And of course an equation of three variables represents a plane in $\mathbb{R}^3$. So, we are looking for planes whose intersection is the given line. Two planes intersect in a line, so two planes should be enough.

Any plane containing the given line must have normal vector which is perpendicular to the direction vector of the line; and in fact any vector perpendicular to the direction vector of the line can be used to define a plane which contains the given line.

So, the normal vectors we are looking for can be taken to be any vectors which are perpendicular to $\begin{bmatrix}-1\\2\\5\end{bmatrix}$. Conveniently, we already found two such vectors in part (a).

Let $\overrightarrow{n}_1 = \begin{bmatrix}2\\1\\0\end{bmatrix}$ and $\overrightarrow{n}_2 = \begin{bmatrix}5\\0\\1\end{bmatrix}$. As commented above, these are perpendicular to the given line, and so define planes parallel to the line. If we insist the planes contain some point we know to be in the line (say $\begin{bmatrix}3\\1\\2\end{bmatrix}$ for instance), then they will define planes that contain the given line. This gives us the equations

$$\begin{aligned}\begin{bmatrix}2\\1\\0\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} &= \begin{bmatrix}2\\1\\0\end{bmatrix}\cdot\begin{bmatrix}3\\1\\2\end{bmatrix}\\ \begin{bmatrix}5\\0\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} &= \begin{bmatrix}5\\0\\1\end{bmatrix}\cdot\begin{bmatrix}3\\1\\2\end{bmatrix}\end{aligned}$$

or

$$\begin{aligned}2x + y \qquad &= 7\\ 5x \qquad + z &= 17\end{aligned}$$

4. Prove the following:

The column vectors of the matrix A are linearly independent	$\iff$	N(A) contains only the zero vector.

Solution: We can rephrase the left side of this implication as:

"The only linear combination of the columns of A which is zero is the one where the coefficients are all zero themselves."

Recalling that a linear combination of column vectors is a matrix-vector product, we can rephrase that as:

"The only vector $\overrightarrow{x}$ for which $A\overrightarrow{x} = \overrightarrow{0}$ is the vector $\overrightarrow{x} = \overrightarrow{0}$."

Of course this is the same as saying:

"The only solution to the system $A\overrightarrow{x} = \overrightarrow{0}$ is $\overrightarrow{x} = \overrightarrow{0}$."

By definition of the null space, this is equivalent to:

"$N(A) = \{\overrightarrow{0}\}$."

So the left and right sides are equivalent.

5. Suppose that for some matrix A, the vector $\overrightarrow{n}$ is an element of $N(A)$ and the vector $\overrightarrow{r}$ is an element of $R(A)$.

Show that $\overrightarrow{n} \cdot \overrightarrow{r} = 0$.

Solution: Let the matrix A have row vectors $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_m$.

$$A = \left(\begin{bmatrix} \text{———} \overrightarrow{v_1} \text{———} \\ \vdots \\ \text{———} \overrightarrow{v_m} \text{———} \end{bmatrix} \right)$$

Since $\overrightarrow{r}$ is in the row space, we know that we must have

$$\overrightarrow{r} = c_1 \overrightarrow{v}_1 + \ldots + c_m \overrightarrow{v}_m$$

We also have $\overrightarrow{n} \in N(A)$, which tells us that $A\overrightarrow{n} = \overrightarrow{0}$, and thus

$$\left(\begin{bmatrix} \text{———} \overrightarrow{v_1} \text{———} \\ \vdots \\ \text{———} \overrightarrow{v_m} \text{———} \end{bmatrix} \right) \begin{bmatrix} | \\ \overrightarrow{n} \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

So $\overrightarrow{n} \cdot \overrightarrow{v}_i = 0$ for all i.

Now we just compute:

$$\begin{aligned} \overrightarrow{n} \cdot \overrightarrow{r} &= \overrightarrow{n} \cdot (c_1 \overrightarrow{v}_1 + \ldots + c_m \overrightarrow{v}_m) \\ &= c_1(\overrightarrow{n} \cdot \overrightarrow{v}_1) + \ldots + c_m(\overrightarrow{n} \cdot \overrightarrow{v}_m) \\ &= c_1(0) + \ldots + c_m(0) \\ &= 0 \end{aligned}$$

6. Suppose that the three *nonzero* vectors $\overrightarrow{x}$, $\overrightarrow{y}$, and $\overrightarrow{z}$ are mutually orthogonal – in other words, $\overrightarrow{x} \cdot \overrightarrow{y} = 0$, $\overrightarrow{x} \cdot \overrightarrow{z} = 0$, and $\overrightarrow{y} \cdot \overrightarrow{z} = 0$.

   Show that these three vectors must be linearly independent.

   Solution: We are given that $\|\overrightarrow{x}\| \neq 0$, $\|\overrightarrow{y}\| \neq 0$, $\|\overrightarrow{z}\| \neq 0$; and we want to show that

$$c_1\overrightarrow{x} + c_2\overrightarrow{y} + c_3\overrightarrow{z} = \overrightarrow{0} \quad \Longrightarrow \quad c_1 = c_2 = c_3 = 0$$

   So, suppose that $c_1\overrightarrow{x} + c_2\overrightarrow{y} + c_3\overrightarrow{z} = \overrightarrow{0}$. In order to try to find a use for the three given dot products, let's take a dot product of both sides with $\overrightarrow{x}$; then with $\overrightarrow{y}$; then with $\overrightarrow{z}$. We get:

$$\begin{aligned} \overrightarrow{x} \cdot (c_1\overrightarrow{x} + c_2\overrightarrow{y} + c_3\overrightarrow{z}) &= \overrightarrow{x} \cdot \overrightarrow{0} \\ \overrightarrow{y} \cdot (c_1\overrightarrow{x} + c_2\overrightarrow{y} + c_3\overrightarrow{z}) &= \overrightarrow{y} \cdot \overrightarrow{0} \\ \overrightarrow{z} \cdot (c_1\overrightarrow{x} + c_2\overrightarrow{y} + c_3\overrightarrow{z}) &= \overrightarrow{z} \cdot \overrightarrow{0} \end{aligned}$$

   Using the three given dot products, these equations turn into:

$$\begin{aligned} c_1\|x\|^2 + 0 + 0 &= 0 \\ 0 + c_2\|y\|^2 + 0 &= 0 \\ 0 + 0 + c_3\|z\|^2 &= 0 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} c_1\|x\|^2 &= 0 \\ c_2\|y\|^2 &= 0 \\ c_3\|z\|^2 &= 0 \end{aligned}$$

   Since we are given that $\|\overrightarrow{x}\| \neq 0$, $\|\overrightarrow{y}\| \neq 0$, $\|\overrightarrow{z}\| \neq 0$, these equations imply that $c_1 = c_2 = c_3 = 0$.

   So we have proved the needed implication, and thus the given vectors must be independent.

Bonus Question: For this problem, we define the following terms:

(a) Two vectors are "compatible" if their dot product is positive or zero; they are "incompatible" if their dot product is negative.

(b) A collection of vectors $\{\overrightarrow{v}_1, \ldots, \overrightarrow{v}_k\}$ is said to be "offensive" if every nonzero vector in $\mathbb{R}^n$ is incompatible with at least one of the vectors $\overrightarrow{v}_i$. (In other words, for every $\overrightarrow{x} \in \mathbb{R}^n$, there is an i such that $\overrightarrow{x}$ and $\overrightarrow{v}_i$ are incompatible.)

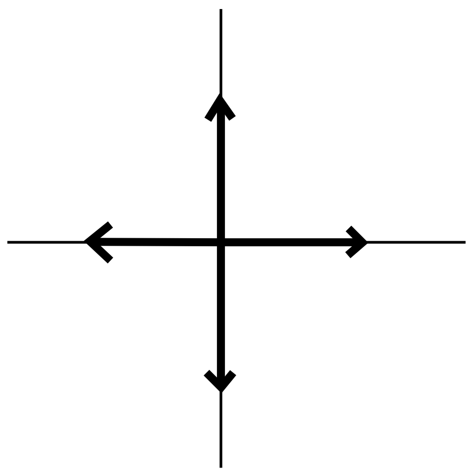


Figure (a)

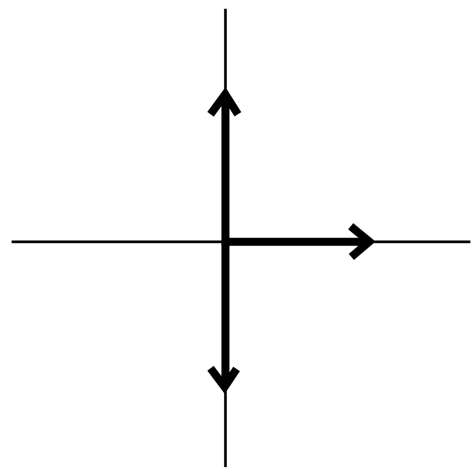


Figure (b)

For example, in Figure (a) above, the four vectors are offensive, because for any vector in $\mathbb{R}^2$, at least one of these four vectors will be incompatible. However, in Figure (b), the three vectors do not form an offensive collection, because there exist vectors like $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ that are compatible with all three of those vectors.

It can be shown that any collection of n or fewer vectors in $\mathbb{R}^n$ cannot be offensive (you may assume this result in your answer to the questions below).

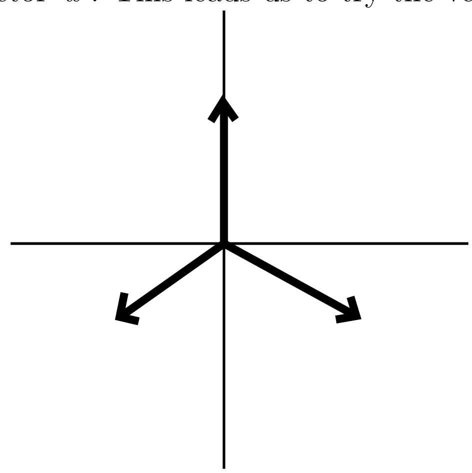
Question 1: What is the *smallest* number of vectors needed to form an offensive set in $\mathbb{R}^2$? Draw a picture representing the vectors in such a collection, and explain your reasoning.

Question 2: What is the *smallest* number of vectors needed to form an offensive set in $\mathbb{R}^n$? Explain your reasoning.

(over)

Question 1 Solution: We are given that it is impossible to form an offensive set of just two vectors in $\mathbb{R}^2$; and the vectors in Figure (a) clearly are an offensive set of four vectors. So, we simply need to determine if it is possible to form an offensive set of three vectors in $\mathbb{R}^2$.

Geometrically, our best chance for this is to try to have the three vectors point as far away from each other as possible, so that hopefully at least one of them will be pointing "backwards" as viewed from any arbitrary vector $\overrightarrow{x}$. This leads us to try the vectors pictured below:



You can convince yourself that these three vectors meet the condition for being offensive by noting that any vector which points "between" two of the vectors will be pointing "backwards" as viewed from the third vector.

Now let's take an algebraic point of view on the problem. Observe that what we have here is actually a system of "linear inequalities", as opposed to our usual system of "linear equations"... In this case, we are looking for vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ such that the system

$$\begin{aligned} a_1x + a_2y &\geq 0 \\ b_1x + b_2y &\geq 0 \\ c_1x + c_2y &\geq 0 \end{aligned}$$

has no nonzero solutions. Expressed in terms of matrices, we can express this by saying

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{“} \geq \text{”} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has no nonzero solutions. (We put the inequality in quotes to recognize that we need all of the components of the product on the left to be ≥ 0.)

One algebraically convenient answer is the following – let the first two rows of the matrix be just the standard basis vectors; this gives us

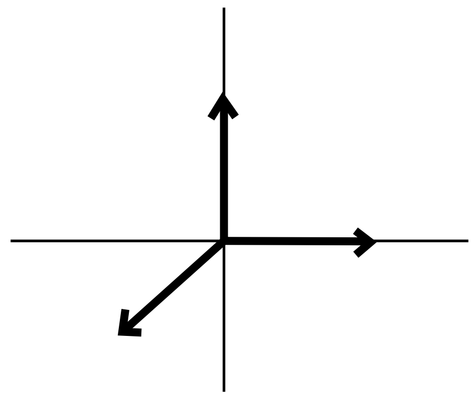
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ c_1 & c_2 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{“} \geq \text{”} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Looking at the first two components, we see that the only possible solutions would have both x and y non-negative. So if we choose c_1 and c_2 to both be negative, then we ensure that no solution is possible. For example, the system

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{“} \geq \text{”} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

clearly cannot possibly have any nonzero solutions.

This matrix corresponds to the three vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$. As pictured below, these are also clearly an offensive set.



(over)

Question 2 Solution: Again, we are given that no collection of n vectors in $\mathbb{R}^n$ can possibly be offensive; so if we can find a collection of $(n+1)$ vectors that is offensive, then we know $(n+1)$ is the smallest number.

Also, we can again translate this into a matrix question. In this case, we want to find a matrix such that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{(n+1)1} & \cdots & & a_{(n+1)n} \end{pmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{``} \geq \text{''} \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

has no nonzero solutions.

The second solution to Question 1 above generalizes nicely to the higher dimensional problem. Following that pattern, we look at

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{``} \geq \text{''} \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Looking at the first n components, we see that we would need for all of the $x_1, \ldots, x_n$ to be non-negative to have any hope for a solution; then of course, as in the previous problem, the final component would necessarily be negative, thus assuring that no solution is possible.

This matrix corresponds to the vectors $e_1, \ldots, e_n, -(e_1 + \cdots + e_n)$, which are therefore an offensive collection of $(n+1)$ vectors. So $(n+1)$ is the smallest possible number.

Equation Sheet

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$