EXAM I

Math 51, Spring 2003.

You have 2 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ______________________________

ID number ____________________

1. __________ (/20 points)

2. __________ (/15 points)

3. __________ (/15 points)

4. __________ (/20 points)

5. __________ (/15 points)

6. __________ (/15 points)

Bonus __________ (/10 points)

Total __________ (/100 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________

Circle your TA's name:

Byoung-du Kim (2 and 6)

Ted Hwa (3 and 7)

Jacob Shapiro (4 and 8)

Ryan Vinroot (A02)

Michel Grueneberg (A03)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Consider the following three vectors:

$$\overrightarrow{u} = \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix} \quad \overrightarrow{v} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \quad \overrightarrow{w} = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$$

(a) Find the magnitude of each of the given vectors.

(b) Find a parametric representation of the unique plane that contains all of the given vectors.

(c) Find an equation representing the unique plane that contains all of the given vectors.

2. Find a parametric representation of the complete set of solutions to the following system of equations.

$$\begin{array}{rcrcrcr} -2x & - & y & + & 5z & = & -5 \\ 3x & + & y & - & 7z & = & 6 \\ 2x & & & - & 4z & = & 2 \end{array}$$

3. (a) Find two linearly independent vectors that are perpendicular to the vector

$$\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

(b) Use your answer from part (a) to find a system of equations for which the following parametric line is the *complete* set of solutions.

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

4. Prove the following:

The column vectors of the matrix A are linearly independent	$\iff$	N(A) contains only the zero vector.

5. Suppose that for some matrix A, the vector $\overrightarrow{n}$ is an element of $N(A)$ and the vector $\overrightarrow{r}$ is an element of $R(A)$.

   Show that $\overrightarrow{n} \cdot \overrightarrow{r} = 0$.

6. Suppose that the three *nonzero* vectors $\overrightarrow{x}$, $\overrightarrow{y}$, and $\overrightarrow{z}$ are mutually orthogonal – in other words, $\overrightarrow{x} \cdot \overrightarrow{y} = 0$, $\overrightarrow{x} \cdot \overrightarrow{z} = 0$, and $\overrightarrow{y} \cdot \overrightarrow{z} = 0$.

   Show that these three vectors must be linearly independent.

Bonus Question: For this problem, we define the following terms:

(a) Two vectors are "compatible" if their dot product is positive or zero; they are "incompatible" if their dot product is negative.

(b) A collection of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is said to be "offensive" if every nonzero vector in $\mathbb{R}^n$ is incompatible with at least one of the vectors $\vec{v}_i$. (In other words, for every $\vec{x} \in \mathbb{R}^n$, there is an i such that $\vec{x}$ and $\vec{v}_i$ are incompatible.)

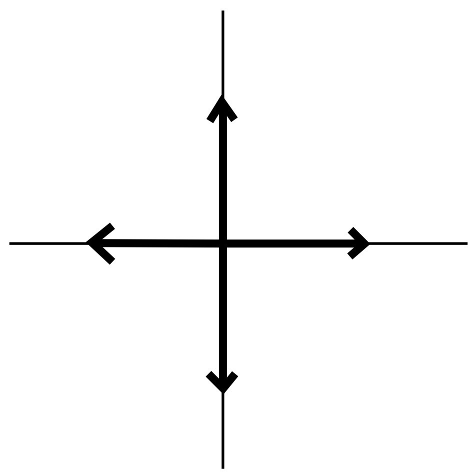

Figure (a)

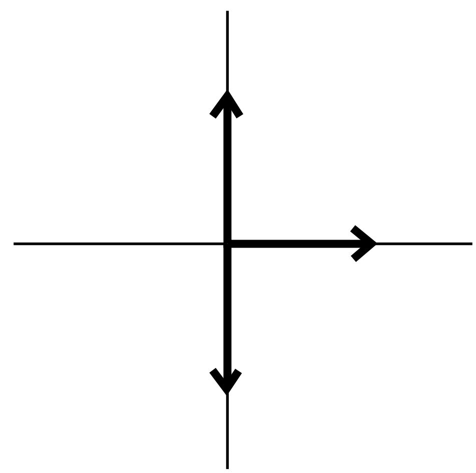

Figure (b)

For example, in Figure (a) above, the four vectors are offensive, because for any vector in $\mathbb{R}^2$, at least one of these four vectors will be incompatible. However, in Figure (b), the three vectors do not form an offensive collection, because there exist vectors like $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ that are compatible with all three of those vectors.

It can be shown that any collection of n or fewer vectors in $\mathbb{R}^n$ cannot be offensive (you may assume this result in your answer to the questions below).

Question 1: What is the *smallest* number of vectors needed to form an offensive set in $\mathbb{R}^2$? Draw a picture representing the vectors in such a collection, and explain your reasoning.

Question 2: What is the *smallest* number of vectors needed to form an offensive set in $\mathbb{R}^n$? Explain your reasoning.

Equation Sheet

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$