

1. (a) Let

$$A = \begin{pmatrix} 7 & -4 \\ 5 & -3 \end{pmatrix}.$$

Is A invertible? If so, find A^{-1} . If not, why not?

Solution:

$\det(A) = 7(-3) - 5(-4) = -1 \neq 0$. Hence A is invertible.

$$A^{-1} = \begin{pmatrix} 3 & -4 \\ 5 & -7 \end{pmatrix}.$$

(b) Let B be the matrix

$$B = \begin{pmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 3 & 0 & p \end{pmatrix}.$$

For what values of p in \mathbb{R} is this matrix invertible?

Solution:

$$\det(B) = 1 \begin{vmatrix} 1 & 0 \\ 0 & p \end{vmatrix} + p \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} = p - 3p = -2p.$$

Since B is invertible whenever $\det(B) \neq 0$, B is invertible for $p \neq 0$.

2. (a) For each of the following transformations, determine whether T IS a linear transformation. For this part only, you do not need to show your work; simply circle “YES” (if it is a linear transformation) or “NO” (if it is not).

$$1. \quad T \begin{pmatrix} x \\ y \end{pmatrix} = xy \quad \text{YES} \quad \text{NO}$$

$$2. \quad T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} |x| \\ 0 \end{pmatrix} \quad \text{YES} \quad \text{NO}$$

$$3. \quad T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x + 2y \\ -3x \end{pmatrix} \quad \text{YES} \quad \text{NO}$$

$$4. \quad T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ x + 1 \\ x + y \end{pmatrix} \quad \text{YES} \quad \text{NO}$$

Solution:

Only transformation #3 is linear.

- (b) For each of the transformations you identified in part (a) as a linear transformation, determine the associated matrix (with respect to the standard basis).

Solution:

For transformation #3:

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Hence the matrix of T with respect to the standard basis is

$$[T] = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ -3 & 0 \end{pmatrix}.$$

3. Let T be a linear transformation from \mathbb{R}^3 to \mathbb{R}^3 so that

$$T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}.$$

(a) What is the matrix A of T relative to the standard basis?

Solution:

$$T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix},$$

Using the linearity of T :

$$T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = T \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}.$$

Similarly:

$$T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - T \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

Hence the matrix of T with respect to the standard basis is

$$[T] = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

(b) Is the matrix A diagonalizable? Why or why not?

Solution:

Yes. It is upper triangular. Hence its eigenvalues are the main diagonal entries, which in this case are 2, 3, 4. Since the eigenvalues of A are distinct it is diagonalizable.

4. You are standing at the point with coordinates $(x, y) = (100, 100)$ on a hillside whose height (in feet above sea level) is given by

$$h(x, y) = 1000 + \frac{1}{100}(x^2 - 3xy + 2y^2)$$

with the positive x -axis pointing to the east and the positive y -axis pointing to the north.

- (a) If you are heading due east, will you initially be ascending or descending? At what angle (in degrees) from the horizontal?

Solution:

Heading east is moving in the positive x -direction. Hence all that we have to do is to compute the partial derivative of $h(x, y)$ with respect to x and evaluate it at $(100, 100)$.

$$\frac{\partial h}{\partial x} = \frac{1}{100}(2x - 3y).$$

Hence $h_x(100, 100) = (200 - 300)/100 = -1$ and you are descending. The angle is 45° .

- (b) If you head due south, will you initially be ascending or descending?

Solution:

Heading south is walking in the negative y -direction. We therefore have to find the sign of $-h_y(100, 100)$.

$$-\frac{\partial h}{\partial y} = -\frac{1}{100}(-3x + 4y).$$

Hence $-h_y(100, 100) = -1$ and you will be descending.

5. (a) Find the matrix A corresponding to the linear transformation that reflects vectors in \mathbb{R}^2 across the line $y = -x$.

Solution:

Let $\{e_1, e_2\}$ denote the standard basis of \mathbb{R}^2 . For this transformation $T(e_1) = -e_2$ and $T(e_2) = -e_1$. Hence

$$[T] = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

- (b) Find the matrix B corresponding to the linear transformation that reflects vectors in \mathbb{R}^2 across the line $y = -x$ and then rotates them by 90° counterclockwise.

Solution:

In this case $T(e_1) = e_1$ and $T(e_2) = -e_2$ so that

$$[T] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- (c) Determine the single geometric transformation represented by B . (A reflection? Then across what line? A rotation? Then by what angle? A projection? Then onto which line?)

Solution:

There are several possible solutions one of which is a reflection about the x -axis.

6. A baseball fan is hurrying up a stadium ramp with hot dog in hand, trying to make it to his seat. His position (in feet) at time t (in minutes) on the ramp is given by

$$\mathbf{r}(t) = \left(\frac{3}{\pi} \cos(2\pi t), \frac{3}{\pi} \sin(2\pi t), 8t \right).$$

- (a) Find his walking speed at time $t = 1$.

Solution:

Denote by $\mathbf{v}(t)$ the velocity vector at time t . Then

$$\mathbf{v}'(t) = (-6 \sin(2\pi t), 6 \cos(2\pi t), 8).$$

Hence $\mathbf{v}(1) = (0, 6, 8)$ and the walking speed

$$\|\mathbf{v}\| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = 10.$$

- (b) Show that for any time t , his acceleration is perpendicular to his velocity.

Solution:

The acceleration at time t is given by

$$\mathbf{a}'(t) = (-12\pi \cos(2\pi t), -12\pi \sin(2\pi t), 0).$$

Hence

$$\mathbf{a} \cdot \mathbf{v} = (-12\pi \cos(2\pi t))(-6 \sin(2\pi t)) + (-12\pi \sin(2\pi t))(6 \cos(2\pi t)) = 0,$$

and the velocity and acceleration are perpendicular to each other.

7. If T is a linear transformation given by

$$T(x_1, x_2, x_3) = \left(3x_1, \frac{3}{\sqrt{2}}(x_2 - x_3), \frac{3}{\sqrt{2}}(x_2 + x_3) \right)$$

and \mathbf{S} is a sphere such that $T(\mathbf{S})$ is a sphere of radius 6, find the radius of \mathbf{S} .

Solution:

Writing down the matrix A corresponding to the linear transformation T , we have

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}.$$

We find $\det(A) = 27$. This is the scaling factor for volume, so

$$\text{Volume}(T(\mathbf{S})) = 27 \cdot \text{Volume}(\mathbf{S}).$$

The formula for the volume of a sphere of radius r is given by $\text{Volume} = \frac{4}{3}\pi r^3$. So $\text{Volume}(T(\mathbf{S})) = \frac{4}{3}\pi 6^3$. Solving for the radius of \mathbf{S} using the above relationship, we find the radius of the original sphere \mathbf{S} is 2.

Alternately, one could note that this transformation is the composition of a scaling transformation and a rotation about the x -axis by 45° . The rotation preserves volume while the scaling transformation scales each component of vectors in \mathbb{R}^3 by a factor of 3, thereby scaling volume by $3^3 = 27$. Hence, this transformation takes spheres to spheres and a radius of length 6 under the transformation must have come from an original radius for S of length 2.