

EXAM II

Math 51, Spring 2002.

You have 2 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name _____

ID number _____

“On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

1. _____ (/30 points)

2. _____ (/30 points)

3. _____ (/30 points)

4. _____ (/30 points)

5. _____ (/30 points)

Bonus _____ (/15 points)

Total _____ (/150 points)

Signature: _____

Circle your TA's name:

Tarn Adams (2 and 6)

Mariel Saez (3 and 7)

Yevgeniy Kovchegov (4 and 8)

Heaseung Kwon (A02)

Alex Meadows (A03)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. (a) Without computing the inverse directly, explain how you know that the matrix A given below is invertible.

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

- (b) Find A^{-1} .

2. For each of the following, either (i) find a matrix (or matrices) satisfying the given descriptions, or (ii) prove that such a matrix (or matrices) cannot exist.

(a) The column space $C(M_1)$ contains the vectors

$$\begin{bmatrix} 4 \\ 3 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 7 \\ 8 \end{bmatrix}$$

and the null space $N(M_1)$ contains the vectors

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$$

(b) The row space $R(M_2)$ contains the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ -3 \\ -4 \end{bmatrix}$$

and the null space $N(M_2)$ contains the vector

$$\begin{bmatrix} 4 \\ 2 \\ 3 \\ 7 \end{bmatrix}$$

- (c) The matrix M_3 is 4×2 , the matrix M_4 is 2×4 , and the 4×4 product M_3M_4 has a 1-dimensional null space.

(Hint: What can you say about the dimension of $N(M_4)$? And what can you say about the relationship between $N(M_4)$ and $N(M_3M_4)$?)

3. We define F_θ to be the linear transformation which flips vectors in the plane \mathbb{R}^2 over the line at angle θ with the x -axis; for example, F_0 flips vectors over the x -axis itself, and $F_{\pi/4}$ flips vectors over the line $y = x$.

Note that we can write this transformation F_θ as the composition of three transformations – first, a clockwise rotation by an angle θ , then a flip over the x -axis, and finally a counterclockwise rotation by an angle θ .

- (a) Derive the matrix for the transformation which flips vectors over the x -axis.

(b) Derive the matrices for the clockwise and counterclockwise rotations described above.

- (c) Use the information from the first two parts to determine the matrix for the transformation F_θ .

4. Prove the following theorems.

- (a) **Thm:** For any linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, there exists a matrix A such that $T(\vec{x}) = A\vec{x}$.

(b) **Thm:** The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if and only if $ad - bc \neq 0$.

5. Let

$$[\vec{v}_1]_{\mathcal{S}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad [\vec{v}_2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and consider the linear transformation T satisfying

$$[T(\vec{v}_1)]_{\mathcal{S}} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \quad [T(\vec{v}_2)]_{\mathcal{S}} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

- (a) It turns out that there exists a basis $\mathcal{B} = \{\vec{w}_1, \vec{w}_2\}$ such that each of the vectors \vec{w}_1, \vec{w}_2 has positive integer components with respect to the standard basis \mathcal{S} , and the matrix $M = [T]_{\mathcal{B}}$ has all integer entries, two of which are zero.

Find $[\vec{w}_1]_{\mathcal{S}}$ and $[\vec{w}_2]_{\mathcal{S}}$, and write down $M = [T]_{\mathcal{B}}$.

(Hint: You should be able to determine reasonable guesses for \vec{w}_1, \vec{w}_2 by inspection, without doing any computations.)

- (b) What is the change of basis matrix C which converts the components of a vector with respect to the basis \mathcal{B} (determined above) into the components of the same vector with respect to the basis \mathcal{S} ?

(c) What is the matrix $A = [T]_{\mathcal{S}}$?

Bonus Question: Prove or find a counterexample to the following claim:

Let A be an invertible matrix such that for each row of A , the elements in that row add up to 1.

Then A^{-1} must have this same property – in other words, for each row of A^{-1} , the elements in that row must add up to 1.

Hint: Consider the vector $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$