EXAM I SOLUTIONS

Math 51, Spring 2002.

You have 2 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ______________________________

ID number____________________

1. __________ (/20 points)

2. __________ (/30 points)

3. __________ (/20 points)

4. __________ (/10 points)

5. __________ (/20 points)

Bonus __________ (/10 points)

Total __________ (/100 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________

Circle your TA's name:

Tarn Adams (2 and 6)

Mariel Saez (3 and 7)

Yevgeniy Kovchegov (4 and 8)

Heaseung Kwon (A02)

Alex Meadows (A03)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Let

$$\vec{u} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

   (a) Compute the length of each of the given vectors, and the dot product for each pair.

   Solution:

$$\begin{aligned} \|\vec{u}\| &= \sqrt{0^2 + 3^2 + 1^2} = \sqrt{10} \\ \|\vec{v}\| &= \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17} \\ \|\vec{w}\| &= \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \end{aligned}$$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (0)(3) + (3)(2) + (1)(2) = 8 \\ \vec{u} \cdot \vec{w} &= (0)(2) + (3)(3) + (1)(4) = 13 \\ \vec{v} \cdot \vec{w} &= (3)(2) + (2)(3) + (2)(4) = 20 \end{aligned}$$

   (b) Let $\theta_{\vec{u},\vec{v}}$ be the angle formed at the origin between the vectors $\vec{u}$ and $\vec{v}$. Evaluate

$$\cos(\theta_{\vec{u},\vec{v}})$$

   Solution:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \|\vec{u}\| \|\vec{v}\| \cos(\theta_{\vec{u},\vec{v}}) \\ 8 &= \sqrt{10}\sqrt{17} \cos(\theta_{\vec{u},\vec{v}}) \\ \frac{8}{\sqrt{170}} &= \cos(\theta_{\vec{u},\vec{v}}) \end{aligned}$$

(c) Find a linear dependence of the three vectors given, or prove that they are independent.

Solution: A linear dependence of the vectors means that there are constants c_1, c_2, c_3 (not all zero) such that

$$c_1\overrightarrow{u} + c_2\overrightarrow{v} + c_3\overrightarrow{w} = \overrightarrow{0}$$

Interpreting the matrix-vector product as a linear combination of column vectors, this is equivalent to having a nontrivial solution to the equation $A\overrightarrow{x} = \overrightarrow{0}$, with

$$A = \begin{pmatrix}0&3&2\\3&2&3\\1&2&4\end{pmatrix}$$

To determine if this is the case, we row reduce the matrix:

$$\begin{pmatrix}0&3&2\\3&2&3\\1&2&4\end{pmatrix} \qquad \begin{pmatrix}1&2&4\\3&2&3\\0&3&2\end{pmatrix}\begin{matrix}r_3\\r_2\\r_1\end{matrix} \qquad \begin{pmatrix}1&2&4\\0&-4&-9\\0&3&2\end{pmatrix}\begin{matrix}r_1\\r_2-3r_1\\r_3\end{matrix}$$

$$\begin{pmatrix}2&0&-1\\0&-4&-9\\0&0&-19\end{pmatrix}\begin{matrix}2r_1+r_2\\r_2\\4r_3+3r_2\end{matrix} \qquad \begin{pmatrix}38&0&0\\0&-76&0\\0&0&-19\end{pmatrix}\begin{matrix}19r_1-r_3\\19r_2-9r_3\\r_3\end{matrix}$$

At this point, we see that the RREF of A will have a pivot in every column, and thus that there will not be any free variables. Thus we conclude that solutions are unique, and so the only solution to $A\overrightarrow{x} = \overrightarrow{0}$ is the zero vector.

So, the only combination of the three given vectors which is zero is the trivial combination; so, the three vectors are linearly independent.

2. (a) Use pivots to prove that a collection of $(n+1)$ vectors in $\mathbb{R}^n$ must have a linear dependence. (Make sure to explain ALL of your reasoning as carefully and clearly as possible.)

Solution: Consider a collection of vectors $\{v_1, \ldots, v_{n+1}\}$ in $\mathbb{R}^n$; determining if they are dependent amounts to finding constants $c_1, \ldots, c_{n+1}$ (not all zero) such that

$$c_1 \overrightarrow{v}_1 + \cdots + c_{n+1} \overrightarrow{v}_{n+1} = \overrightarrow{0}$$

If we let A be the matrix whose column vectors are the given vectors above, then this amounts to finding a non-trivial solution to the matrix equation $A\overrightarrow{x} = \overrightarrow{0}$.

Of course, this matrix has only n rows (since each column is a vector in $\mathbb{R}^n$) – so, since there can be at most one pivot in each row of RREF(A), we conclude that there are at most n pivots.

However RREF(A) has $(n+1)$ columns, since there are that many of the original vectors – therefore, at least one column in RREF(A) must be without a pivot, and thus we must have at least one free variable in our system of equations.

Since we may choose any value we like for this free variable, we can in particular choose a non-zero value. This means we have a non-trivial solution to our system, and thus that there is a linear dependence among our original $(n+1)$ vectors.

(b) Prove that if the system of equations represented by

$$A\overrightarrow{x} = \overrightarrow{b} \qquad\qquad \text{where } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix}$$

has a solution, then we can conclude that $\overrightarrow{b}$ is in the column space of A.

Solution: Let the column vectors of A be the vectors $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_n$.

We are given that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution; this means that there is some vector

$$\overrightarrow{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

satisfying that equation.

If we interpret the matrix vector product as a linear combination of column vectors, this means that

$$x_1\overrightarrow{v_1} + \ldots + x_n\overrightarrow{v_n} = \overrightarrow{b}$$

So $\overrightarrow{b}$ is a linear combination of column vectors of A, and thus is in the column space of A.

3. (a) Find the null space for the matrix below.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

Solution: First we row reduce the given matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 - r_1 \end{matrix} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} r_1 - r_2 \\ r_2 \\ r_3 + r_2 \end{matrix}$$

The null space is the set of solutions to RREF$(A) = 0$. We have two pivot variables (x_1 and x_2) and one free variable (x_3); solving for the pivot variables in terms of the free variables, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

So the null space is

$$N(A) = \left\{ x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(b) Use your result from part (a) to find a parametric representation of the solution set to the system of equations below WITHOUT row reducing; explain how you know your answer is complete.

$$\begin{array}{ccccccc} x & + & y & & & = & 1 \\ & & y & + & z & = & 0 \\ x & & & - & z & = & 1 \end{array}$$

Solution: First, note that the vector $x_p = (1, 0, 0)$ is a solution to the given system.

Also, we have the theorem presented in class saying that in any system of equations, the complete solution set is given by

$$x_p + N(A)$$

Since we know the null space from the previous problem, we conclude that the complete solution set is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

4. Use the Cauchy-Schwarz inequality

$$|\overrightarrow{v} \cdot \overrightarrow{w}| \leq \|\overrightarrow{v}\|\|\overrightarrow{w}\|$$

to prove the Triange Inequality:

$$\|\overrightarrow{v} + \overrightarrow{w}\| \leq \|\overrightarrow{v}\| + \|\overrightarrow{w}\|$$

(Hint: Begin by computing $\|\overrightarrow{v} + \overrightarrow{w}\|^2$ with dot products, and then plug in the Cauchy-Schwarz inequality when the opportunity arises.)

Solution:

$$\begin{aligned}
\|\overrightarrow{v} + \overrightarrow{w}\|^2 &= (\overrightarrow{v} + \overrightarrow{w}) \cdot (\overrightarrow{v} + \overrightarrow{w}) \\
&= \overrightarrow{v} \cdot \overrightarrow{v} + \overrightarrow{v} \cdot \overrightarrow{w} + \overrightarrow{w} \cdot \overrightarrow{v} + \overrightarrow{w} \cdot \overrightarrow{w} \\
&= \|\overrightarrow{v}\|^2 + 2\,(\overrightarrow{w} \cdot \overrightarrow{v}) + \|\overrightarrow{w}\|^2 \\
&\leq \|\overrightarrow{v}\|^2 + 2\|\overrightarrow{v}\|\|\overrightarrow{w}\| + \|\overrightarrow{w}\|^2 \qquad \text{(by the C.S. inequality)} \\
&\leq (\|\overrightarrow{v}\| + \|\overrightarrow{w}\|)^2
\end{aligned}$$

Taking the square root of both sides, we conclude that

$$\|\overrightarrow{v} + \overrightarrow{w}\| \leq \|\overrightarrow{v}\| + \|\overrightarrow{w}\|$$

5. The matrix A below has the given reduced row echelon form (You do not need to verify this).

$$A = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 3 & 2 & 3 \\ 3 & 2 & 2 & 6 \\ 4 & 1 & 2 & 7 \end{pmatrix} \qquad \operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & \frac{2}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Using this information, write down bases for the null space, column space, and row space of A.

Solution: As proved in class, we know that a basis for $R(A)$ is given by the non-zero row vectors in RREF(A); so, we have

$$\text{Basis for } R(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{2}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Similarly, we know that a basis for $C(A)$ is given by the column vectors of our original matrix A corresponding to pivots in RREF(A); so, we have

$$\text{Basis for } C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \\ 7 \end{bmatrix} \right\}$$

To find a basis for $N(A)$, we solve the homogenous system:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5}x_3 \\ -\frac{2}{5}x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{2}{5} \\ -\frac{2}{5} \\ 1 \\ 0 \end{bmatrix}$$

This is a parametric representation of $N(A)$, and we know from class that the direction vector(s) form a basis; so,

$$\text{Basis for } N(A) = \left\{ \begin{bmatrix} -\frac{2}{5} \\ -\frac{2}{5} \\ 1 \\ 0 \end{bmatrix} \right\}$$

Bonus Question: Let the function f project vectors in $\mathbb{R}^3$ to the xy-plane; in other words,

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Also suppose that the three vectors

$$\overrightarrow{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \overrightarrow{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \overrightarrow{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

are all perpendicular to each other.

Show that it is NOT possible for all three of the angles created at the origin by the vectors $f(\overrightarrow{u}), f(\overrightarrow{v}), f(\overrightarrow{w})$ to be acute.

(Hint: Try phrasing the given conditions on the three vectors $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^3$ and the three desired conditions on the vectors $f(\overrightarrow{u}), f(\overrightarrow{v}), f(\overrightarrow{w})$ in terms of dot products, and then expand those expressions in terms of components.)

Solution: Recall that two vectors are perpendicular if and only if their dot product is zero; also, given that

$$\overrightarrow{u} \cdot \overrightarrow{v} = \|\overrightarrow{u}\|\|\overrightarrow{v}\| \cos(\theta_{\overrightarrow{u},\overrightarrow{v}})$$

we note that the angle between two vectors is acute if and only if their dot product is positive.

Using this, we see that we are given

$$u_1v_1 + u_2v_2 + u_3v_3 = 0 \qquad u_1w_1 + u_2w_2 + u_3w_3 = 0 \qquad v_1w_1 + v_2w_2 + v_3w_3 = 0$$

and then need to show that it is not possible to have

$$u_1v_1 + u_2v_2 > 0 \qquad u_1w_1 + u_2w_2 > 0 \qquad v_1w_1 + v_2w_2 > 0$$

To show this, let us assume the contrary – namely, that the six equations above are all true; from this, we will derive a contradiction, as follows:

Subtracting the bottom equations from the top equations, we conclude that

$$u_3v_3 < 0 \qquad u_3w_3 < 0 \qquad v_3w_3 < 0$$

If all three of the quantities above are negative, then their product must be also; but their product is

$$(u_3v_3)(u_3w_3)(v_3w_3) = u_3^2v_3^2w_3^2$$

which is a square, and thus cannot be negative. This is a contradiction.