EXAM I

Math 51, Spring 2002.

You have 2 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ______________________________

ID number____________________

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________

1. __________ (/20 points)

2. __________ (/30 points)

3. __________ (/20 points)

4. __________ (/10 points)

5. __________ (/20 points)

Bonus __________ (/10 points)

Total __________ (/100 points)

Circle your TA's name:

Tarn Adams (2 and 6)

Mariel Saez (3 and 7)

Yevgeniy Kovchegov (4 and 8)

Heaseung Kwon (A02)

Alex Meadows (A03)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Let

$$\overrightarrow{u} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad \overrightarrow{v} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \quad \overrightarrow{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

(a) Compute the length of each of the given vectors, and the dot product for each pair.

(b) Let $\theta_{\overrightarrow{u},\overrightarrow{v}}$ be the angle formed at the origin between the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$. Evaluate

$$\cos(\theta_{\overrightarrow{u},\overrightarrow{v}})$$

(c) Find a linear dependence of the three vectors given, or prove that they are independent.

2. (a) Use pivots to prove that a collection of $(n+1)$ vectors in $\mathbb{R}^n$ must have a linear dependence. (Make sure to explain ALL of your reasoning as carefully and clearly as possible.)

(b) Prove that if the system of equations represented by

$$A\overrightarrow{x} = \overrightarrow{b} \qquad\qquad \text{where } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix}$$

has a solution, then we can conclude that $\overrightarrow{b}$ is in the column space of A.

3. (a) Find the null space for the matrix below.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

(b) Use your result from part (a) to find a parametric representation of the solution set to the system of equations below WITHOUT row reducing; explain how you know your answer is complete.

$$\begin{array}{ccccccc} x & + & y & & & = & 1 \\ & & y & + & z & = & 0 \\ x & & & - & z & = & 1 \end{array}$$

4. Use the Cauchy-Schwarz inequality

$$|\overrightarrow{v} \cdot \overrightarrow{w}| \leq \|\overrightarrow{v}\|\|\overrightarrow{w}\|$$

to prove the Triange Inequality:

$$\|\overrightarrow{v} + \overrightarrow{w}\| \leq \|\overrightarrow{v}\| + \|\overrightarrow{w}\|$$

(Hint: Begin by computing $\|\overrightarrow{v} + \overrightarrow{w}\|^2$ with dot products, and then plug in the Cauchy-Schwarz inequality when the opportunity arises.)

5. The matrix A below has the given reduced row echelon form (You do not need to verify this).

$$A = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 3 & 2 & 3 \\ 3 & 2 & 2 & 6 \\ 4 & 1 & 2 & 7 \end{pmatrix} \qquad \text{rref}(A) = \begin{pmatrix} 1 & 0 & \frac{2}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Using this information, write down bases for the null space, column space, and row space of A.

Bonus Question: Let the function f project vectors in $\mathbb{R}^3$ to the xy-plane; in other words,

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Also suppose that the three vectors

$$\overrightarrow{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \overrightarrow{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \overrightarrow{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

are all perpendicular to each other.

Show that it is NOT possible for all three of the angles created at the origin by the vectors $f(\overrightarrow{u}), f(\overrightarrow{v}), f(\overrightarrow{w})$ to be acute.

(Hint: Try phrasing the given conditions on the three vectors $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^3$ and the three desired conditions on the vectors $f(\overrightarrow{u}), f(\overrightarrow{v}), f(\overrightarrow{w})$ in terms of dot products, and then expand those expressions in terms of components.)