

# FINAL EXAM

Math 51, Spring 2002.

You have 3 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING  
TO RECEIVE CREDIT

Good luck!

Name \_\_\_\_\_

ID number \_\_\_\_\_

1. \_\_\_\_\_ (/40 points)

2. \_\_\_\_\_ (/40 points)

3. \_\_\_\_\_ (/40 points)

4. \_\_\_\_\_ (/30 points)

5. \_\_\_\_\_ (/30 points)

6. \_\_\_\_\_ (/20 points)

Bonus \_\_\_\_\_ (/20 points)

Total \_\_\_\_\_ (/200 points)

“On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

Signature: \_\_\_\_\_

Circle your TA's name:

Tarn Adams (2 and 6)

Mariel Saez (3 and 7)

Yevgeniy Kovchegov (4 and 8)

Heaseung Kwon (A02)

Alex Meadows (A03)

Circle your section meeting time:

11:00am    1:15pm    7pm

1. Suppose that  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  are given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 y \\ y^2 - x^2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \quad g\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} \sin u \\ \cos u \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

(a) Evaluate  $J_{f, \vec{a}}$  (where  $\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix}$ ) and  $J_{g, f(\vec{a})}$ , in terms of  $x$  and  $y$ .

(b) *Without* computing the composition function  $g \circ f$ , evaluate  $J_{g \circ f, \vec{a}}$ , in terms of  $x$  and  $y$ .

- (c) Using the result from part (b), determine  $\frac{\partial s}{\partial y}$  *without* explicitly computing  $s$  as a function of  $x$  and  $y$ . Explain your reasoning.

2. (a) Write out the single variable limit that defines the directional derivative  $D_{\vec{v}}f(\vec{a})$ .

- (b) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 - 3xy^2$$

Compute the directional derivative of  $f$  at the point  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  in the direction  $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$  directly from the definition above.

- (c) Write out the definition (involving a multivariable limit) of the derivative transformation  $D_{f, \vec{a}}$  of a differentiable function  $f$ .

- (d) Suppose that a function  $g$  has

$$D_{\vec{v}_1} f(\vec{a}) = \vec{w}_1 \quad D_{\vec{v}_2} f(\vec{a}) = \vec{w}_2 \quad D_{\vec{v}_1 + \vec{v}_2} f(\vec{a}) = \vec{w}_1 + 2\vec{w}_2$$

for some nonzero vectors  $\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2$ .

Show that  $g$  cannot be differentiable, making sure to be clear about all the steps in your argument. (Hint: Think about the relationship between  $D_{f, \vec{a}}(\vec{v})$  and  $D_{\vec{v}} f(\vec{a})$ ).

3. A linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to “preserve angles” if it is one-to-one and, for all vectors  $\vec{v}$  and  $\vec{w}$ , the angle between  $\vec{v}$  and  $\vec{w}$  is equal to the angle between  $T(\vec{v})$  and  $T(\vec{w})$ .

It can be shown that a linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  preserves angles if and only if

$$\|T(\vec{e}_1)\| = \|T(\vec{e}_2)\| \neq 0 \quad \text{and} \quad T(\vec{e}_1) \cdot T(\vec{e}_2) = 0$$

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be “conformal” if for all points  $\vec{a}$  where the derivative transformation is not identically zero, the derivative transformation  $D_{f,\vec{a}}$  preserves angles.

- (a) Show that the function

$$f \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$$

is conformal.

(b) Prove or find a counterexample to the following:

**Claim:** The composition of two conformal functions must be conformal.

(c) Use parts (a) and (b) to determine if the function

$$h \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x^4 - 6x^2y^2 + y^4 \\ 4x^3y - 4xy^3 \end{bmatrix}$$

is conformal. (DO NOT compute the derivative transformation for  $h$ .)

4. Find all critical points and the maximum and minimum values of the function

$$f \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = (1 - x^2 - y^2 - z^2)(x + 1)$$

on the solid unit ball defined by  $x^2 + y^2 + z^2 \leq 1$ . (Hint: you should be able to avoid using Lagrange multipliers on the boundary by making certain observations about  $f$ .)

5. Consider the function

$$f \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = x^3 + xy + y^3$$

(a) Find all critical points of this function.

- (b) Show that one of the critical points from part (a) is a saddle point. (Hint: in reference to a critical point  $\begin{bmatrix} a \\ b \end{bmatrix}$ , consider  $f\left(\begin{bmatrix} a+h \\ b \end{bmatrix}\right) - f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$ , and use this to show the critical point satisfies the definition of saddle point.)

6. Consider the function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 9(x+1)^5 y^3 + \sin\left(\frac{\pi y^2}{2}\right) + x^3 y - e^{xy}$$

restricted to the domain  $D$  defined by  $x^6 + y^6 \leq 1$ .

Show that NEITHER the maximum NOR the minimum value of  $f$  on  $D$  can be attained at the point  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Make sure to explain your reasoning.

**Bonus Question:** Show that if  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is conformal and if  $\vec{a}$  is a point such that  $D_{g, \vec{a}}$  is not identically zero, then the vectors

$$\nabla g_1(\vec{a}), \nabla g_2(\vec{a}), \dots, \nabla g_n(\vec{a})$$

must be independent.