EXAM II SOLUTIONS

Math 51, Spring 2001.

You have 2 hours.

No notes, no books.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ____________________________________

ID number________________________

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ________________________

1. ____________ (/20 points)

2. ____________ (/20 points)

3. ____________ (/20 points)

4. ____________ (/20 points)

5. ____________ (/20 points)

Bonus ____________ (/10 points)

Total ____________ (/100 points)

Circle your TA's name:

Kuan Ju Liu (2 and 6)

Robert Sussland (3 and 7)

Hunter Tart (4 and 8)

Alex Meadows (10)

Dana Rowland (11)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. (a) Use determinants to find the area of the triangle in $\mathbb{R}^2$ with vertices located at

$$\begin{bmatrix}1\\5\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}$$

Solution: The triangle determined by those three points is exactly half of the parallelogram that is determined by any two sides. These sides can be represented as vectors, with tails on one of the given vectors and heads on the other two. For example,

$$\vec{v}_1 = \begin{bmatrix}1\\5\end{bmatrix} - \begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}-3\\2\end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix}-2\\1\end{bmatrix} - \begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}-6\\-2\end{bmatrix}$$

The area of the parallelogram is then

$$\det \begin{bmatrix}-3 & -6\\2 & -2\end{bmatrix} = (-3)(-2) - (-6)(2) = 18$$

So, the area of the triangle is then 9.

(b) Use the cross product to determine the area of the triangle in $\mathbb{R}^3$ with vertices located at

$$\begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}2\\5\\3\end{bmatrix}, \begin{bmatrix}0\\-4\\2\end{bmatrix}$$

Solution: As above, the triangle here is exactly half of the parallelogram determined by

$$\vec{v}_1 = \begin{bmatrix}1\\2\\4\end{bmatrix} - \begin{bmatrix}0\\-4\\2\end{bmatrix} = \begin{bmatrix}1\\6\\2\end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix}2\\5\\3\end{bmatrix} - \begin{bmatrix}0\\-4\\2\end{bmatrix} = \begin{bmatrix}2\\9\\1\end{bmatrix}$$

This parallelogram has area equal to the length of the cross product of these two vectors.

$$\vec{v}_1 \times \vec{v}_2 = \det\left(\begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\1 & 6 & 2\\2 & 9 & 1\end{bmatrix}\right) = \begin{bmatrix}-12\\3\\-3\end{bmatrix}$$

So $|\vec{v}_1 \times \vec{v}_2| = \sqrt{12^2+3^2+3^2} = \sqrt{162} = 9\sqrt{2}$, and thus we conclude that the area of the triangle is half of that, or $9/\sqrt{2}$

(c) Noticing that for vectors $\overrightarrow{v}$, $\overrightarrow{w}$, and $\overrightarrow{x}$, we have

$$(\overrightarrow{v} \times \overrightarrow{w}) \cdot \overrightarrow{x} = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

use the properties of the determinant to show that the cross product of two vectors is always perpendicular to each of those two vectors.

Solution: To show that the cross product is perpendicular to $\overrightarrow{v}$, we simply compute the dot product of those two vectors, using the observed formula above.

$$(\overrightarrow{v} \times \overrightarrow{w}) \cdot \overrightarrow{v} = \det \begin{pmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

Since two of the rows of this matrix are identical, we immediately conclude that this determinant is zero, and so the dot product in question is zero. So, the cross product is perpendicular to $\overrightarrow{v}$.

A similar calculation shows that the cross product is perpendicular to $\overrightarrow{w}$.

2. Let the basis $\mathcal{B}$ be given by the vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$, with

$$[\overrightarrow{v_1}]_{\mathcal{S}} = \begin{bmatrix} -2/7 \\ 3/7 \\ 6/7 \end{bmatrix}, \quad [\overrightarrow{v_2}]_{\mathcal{S}} = \begin{bmatrix} 6/7 \\ -2/7 \\ 3/7 \end{bmatrix}, \quad [\overrightarrow{v_3}]_{\mathcal{S}} = \begin{bmatrix} 3/7 \\ 6/7 \\ -2/7 \end{bmatrix}$$

(Note that the vectors in $\mathcal{B}$ are all orthogonal, and are all unit vectors.)

(a) Find the matrix C which converts from $\mathcal{B}$ coordinates to $\mathcal{S}$ (standard basis) coordinates.

Solution: As was shown in class, the matrix C referred to here has columns which are just the $\mathcal{S}$ representations of the vectors in $\mathcal{B}$. So, the matrix is just

$$C = \begin{pmatrix} -2/7 & 6/7 & 3/7 \\ 3/7 & -2/7 & 6/7 \\ 6/7 & 3/7 & -2/7 \end{pmatrix}$$

(b) Let T be the linear transformation which rotates vectors in $\mathbb{R}^3$ by an angle of $\pi/6$ radians around $\overrightarrow{v_1}$, in the direction from $\overrightarrow{v_2}$ toward $\overrightarrow{v_3}$. What is the matrix M for T with respect to the basis $\mathcal{B}$?

Solution: The description of T above tells us the following:

$$T(\overrightarrow{v_1}) = \overrightarrow{v_1} \qquad T(\overrightarrow{v_2}) = \frac{\sqrt{3}}{2}\overrightarrow{v_2} + \frac{1}{2}\overrightarrow{v_3} \qquad T(\overrightarrow{v_3}) = \frac{-1}{2}\overrightarrow{v_2} + \frac{\sqrt{3}}{2}\overrightarrow{v_3}$$

Converting each of these vectors into $\mathcal{B}$ coordinates, we get

$$[T(\overrightarrow{v_1})]_{\mathcal{B}} = [\overrightarrow{v_1}]_{\mathcal{B}} \qquad [T(\overrightarrow{v_2})]_{\mathcal{B}} = \left[\frac{\sqrt{3}}{2}\overrightarrow{v_2} + \frac{1}{2}\overrightarrow{v_3}\right]_{\mathcal{B}} \qquad [T(\overrightarrow{v_3})]_{\mathcal{B}} = \left[\frac{-1}{2}\overrightarrow{v_2} + \frac{\sqrt{3}}{2}\overrightarrow{v_3}\right]_{\mathcal{B}}$$

$$M\,[\overrightarrow{v_1}]_{\mathcal{B}} = M\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad M\,[\overrightarrow{v_2}]_{\mathcal{B}} = M\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{bmatrix} \qquad M\,[\overrightarrow{v_3}]_{\mathcal{B}} = M\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\\frac{-1}{2}\\\frac{\sqrt{3}}{2}\end{bmatrix}$$

So the matrix M is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(c) Let A be the matrix for T (the transformation from part (b)) with respect to the standard basis $\mathcal{S}$. Express A in terms of M and C. (You do not need to explicitly compute A.) Explain.

Solution: We have the diagram below:

$$\begin{array}{ccc} [\overrightarrow{v}]_{\mathcal{S}} & \xrightarrow{A} & \left[\overrightarrow{T(v)}\right]_{\mathcal{S}} \\ C \uparrow\downarrow C^{-1} & & C \uparrow\downarrow C^{-1} \\ [\overrightarrow{v}]_{\mathcal{B}} & \xrightarrow{M} & \left[\overrightarrow{T(v)}\right]_{\mathcal{B}} \end{array}$$

From this diagram we see that the matrix A can be reproduced by first applying C^{-1}, then M, and then C. So,

$$A = C\, M\, C^{-1}$$

(d) Let F be the composition transformation defined by $F = R \circ T$, where R is the transformation which rotates a vector by an angle of $\pi/6$ radians around $\overrightarrow{e_1}$, in the direction from $\overrightarrow{e_2}$ toward $\overrightarrow{e_3}$. What is the matrix B for the transformation F with respect to the standard basis $\mathcal{S}$? (Express your answer in terms of M and C.)

Solution: We already have the matrix for T with respect to the standard basis; so, we just need to find the matrix for the rotation R; we will call it D. In fact, this computation is almost identical to the one in part (b), and in fact yields the same answer.

The description above tells us the following:

$$R(\overrightarrow{e_1}) = \overrightarrow{e_1} \qquad R(\overrightarrow{e_2}) = \frac{\sqrt{3}}{2}\overrightarrow{e_2} + \frac{1}{2}\overrightarrow{e_3} \qquad R(\overrightarrow{e_3}) = \frac{-1}{2}\overrightarrow{e_2} + \frac{\sqrt{3}}{2}\overrightarrow{e_3}$$

Converting each of these vectors into $\mathcal{S}$ coordinates, we get

$$[R(\overrightarrow{e_1})]_{\mathcal{S}} = [\overrightarrow{e_1}]_{\mathcal{S}} \qquad [R(\overrightarrow{e_2})]_{\mathcal{S}} = \left[\frac{\sqrt{3}}{2}\overrightarrow{e_2} + \frac{1}{2}\overrightarrow{e_3}\right]_{\mathcal{S}} \qquad [R(\overrightarrow{e_3})]_{\mathcal{S}} = \left[\frac{-1}{2}\overrightarrow{e_2} + \frac{\sqrt{3}}{2}\overrightarrow{e_3}\right]_{\mathcal{S}}$$

$$D\,[\overrightarrow{e_1}]_{\mathcal{S}} = D\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad D\,[\overrightarrow{e_2}]_{\mathcal{S}} = D\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{bmatrix} \qquad D\,[\overrightarrow{e_3}]_{\mathcal{S}} = D\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\\frac{-1}{2}\\\frac{\sqrt{3}}{2}\end{bmatrix}$$

So the matrix D is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

which of course is equal to M.

So, since $F = R \circ T$, and T has matrix $A = C\,M\,C^{-1}$ in standard coordinates, we conclude that the matrix for F in standard coordinates is

$$B = D\,A = D\,C\,M\,C^{-1} = M\,C\,M\,C^{-1}$$

3. (a) Compute the matrix product AB where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 6 & -2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 5 & 2 \\ 3 & -2 & 1 \\ 4 & 3 & 1 \end{pmatrix}$$

Solution:

$$AB = \begin{pmatrix} 29 & 11 & 9 \\ 18 & 34 & 10 \end{pmatrix}$$

(b) Let A be given by the 2×3 matrix below, and let B be the $3 \times n$ matrix with rows $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \qquad B = \left(\begin{bmatrix} \text{———} \overrightarrow{v_1} \text{———} \\ \text{———} \overrightarrow{v_2} \text{———} \\ \text{———} \overrightarrow{v_3} \text{———} \end{bmatrix}\right)$$

Write the row vectors of the product AB as linear combinations of the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$.

Solution: Let's write the product AB as

$$AB = \left(\begin{bmatrix} \text{———} \overrightarrow{w_1} \text{———} \\ \text{———} \overrightarrow{w_2} \text{———} \end{bmatrix}\right)$$

The first component of $\overrightarrow{w_1}$ is the dot product of the top row of A with the first column of B; in other words, it is a linear combination of the numbers in the first column of B, with coefficients given by a_1, a_2, a_3.

A similar statement is true of all of the components of $\overrightarrow{w_1}$ – the ith component of $\overrightarrow{w_1}$ is a linear combination of the numbers in the ith column of B, with coefficients given by a_1, a_2, a_3.

So each of the components of $\overrightarrow{w_1}$ is a linear combination of the corresponding components of $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$, with coefficients given by a_1, a_2, a_3.

So, the whole vector $\overrightarrow{w_1}$ is in fact just a linear combination of the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$, with coefficients given by a_1, a_2, a_3.

$$\overrightarrow{w_1} = a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + a_3\overrightarrow{v_3}$$

Similarly,

$$\overrightarrow{w_2} = a_4\overrightarrow{v_1} + a_5\overrightarrow{v_2} + a_6\overrightarrow{v_3}$$

4. Let the linear transformations below have matrices A, B, L, M, with domains and ranges as described in the diagram below:

$$\mathbb{R}^4 \quad \overset{A}{\longrightarrow} \quad \mathbb{R}^2 \quad \overset{B}{\longrightarrow} \quad \mathbb{R}^3 \quad \overset{L}{\longrightarrow} \quad \mathbb{R}^1 \quad \overset{M}{\longrightarrow} \quad \mathbb{R}^3$$

For the following problems, you might want to consider using the Rank-Nullity Theorem:

(a) Find the largest possible dimension for $C(A)$

Solution: $C(A)$ must be a subspace of $\mathbb{R}^2$, so its dimension can be at most 2. For example, A could be the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(b) Find the smallest possible dimension for $N(BA)$

Solution: We know that $N(BA) \supset N(A)$ (since $A\overrightarrow{x} = 0 \Rightarrow BA\overrightarrow{x} = 0$). And the dimension of $N(A)$ must be at least 2, since from part (a) we know that the dimension of $C(A)$ can be at most 2.

So, $N(BA)$ has dimension at least 2. For example, using the same matrix A as in part (a), we could have

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad BA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) Find the smallest possible dimension for $N(MLB)$

Solution: We know that $C(MLB) \subset C(M)$; and by the Rank-Nullity Theorem, the dimension of $C(M)$ is at most 1. So, the dimension of $C(MLB)$ is at most one. Again applying the Rank-Nullity Theorem, we conclude that $N(MLB)$ must have dimension at least 1.

(d) Suppose that $N(B)$ has dimension 1; what are the possible dimensions of $C(MLBA)$? Explain.

Solution: From part (c), we know that the dimension of $N(MLB)$ is at least 1; so, we conclude that the dimension of $C(MLB)$ is at most 1. Since $C(MLBA) \subset C(MLB)$, we know that $C(MLBA)$ is also at most 1.

So the only possible dimensions of $C(MLBA)$ are 0 or 1. We need at this point to check only that these are not inconsistent with the given condition that the dimension of $N(B)$ is 1. We see this by merely providing examples of matrices A, B, L, M that meet the desired criteria:

Case 1:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \qquad MLBA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \dim(C(MLBA)) = 1$$

Case 2:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \qquad MLBA = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \dim(C(MLBA)) = 0$$

So, 0 and 1 are the possible values for the dimension of $C(MLBA)$.

5. Prove that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if and only if the determinant $ad - bc$ is not equal to zero.

Solution:

($\Leftarrow$): If the determinant $ad - bc$ is nonzero, then we easily check that

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is the inverse of A. So, A is invertible.

($\Rightarrow$): We will prove the contrapositive; in particular, we will assume that $ad - bc = 0$, and then show that in that case, A is not invertible. We will accomplish this by demonstrating that the columns of A are dependent.

First, we consider the case where a and b are both zero. Clearly the rref(A) has only one pivot, so it is not invertible.

Now we proceed assuming that at least one of a and b is nonzero.

Given that $ad - bc = 0$, we observe that of course $ad = bc$. Then

$$b \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} ab \\ bc \end{bmatrix} = \begin{bmatrix} ab \\ ad \end{bmatrix} = a \begin{bmatrix} b \\ d \end{bmatrix}$$

So, we see that at least one of the column vectors is a multiple of the other one. So, the column vectors of A are dependent, so A must not be invertible, as desired.

Bonus Question– Prove or find a counterexample to the following statement:

Proposition: If an $n \times n$ matrix A has the property that

$$A^2 = 0_n$$

(where 0_n is the $n \times n$ matrix whose entries are all zero), then the matrix A must equal 0_n.

Solution: The proposition above is **FALSE**. An easy counterexample is the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

It is easy to check that $A^2 = 0_2$, but of course A itself is clearly not equal to 0_2!

In general, any $n \times n$ matrix A with $C(A) \subset N(A)$ will be a counterexample.