EXAM II

Math 51, Spring 2001.

You have 2 hours.

No notes, no books.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ______________________________

ID number____________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

5. __________ (/20 points)

Bonus __________ (/10 points)

Total __________ (/100 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________

Circle your TA's name:

Kuan Ju Liu (2 and 6)

Robert Sussland (3 and 7)

Hunter Tart (4 and 8)

Alex Meadows (10)

Dana Rowland (11)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. (a) Use determinants to find the area of the triangle in $\mathbb{R}^2$ with vertices located at

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(b) Use the cross product to determine the area of the triangle in $\mathbb{R}^3$ with vertices located at

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}$$

(c) Noticing that for vectors $\overrightarrow{v}$, $\overrightarrow{w}$, and $\overrightarrow{x}$, we have

$$(\overrightarrow{v} \times \overrightarrow{w}) \cdot \overrightarrow{x} = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

use the properties of the determinant to show that the cross product of two vectors is always perpendicular to each of those two vectors.

2. Let the basis $\mathcal{B}$ be given by the vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$, with

$$[\overrightarrow{v_1}]_{\mathcal{S}} = \begin{bmatrix} -2/7 \\ 3/7 \\ 6/7 \end{bmatrix}, \quad [\overrightarrow{v_2}]_{\mathcal{S}} = \begin{bmatrix} 6/7 \\ -2/7 \\ 3/7 \end{bmatrix}, \quad [\overrightarrow{v_3}]_{\mathcal{S}} = \begin{bmatrix} 3/7 \\ 6/7 \\ -2/7 \end{bmatrix}$$

(Note that the vectors in $\mathcal{B}$ are all orthogonal, and are all unit vectors.)

(a) Find the matrix C which converts from $\mathcal{B}$ coordinates to $\mathcal{S}$ (standard basis) coordinates.

(b) Let T be the linear transformation which rotates vectors in $\mathbb{R}^3$ by an angle of $\pi/6$ radians around $\overrightarrow{v_1}$, in the direction from $\overrightarrow{v_2}$ toward $\overrightarrow{v_3}$. What is the matrix M for T with respect to the basis $\mathcal{B}$?

(c) Let A be the matrix for T (the transformation from part (b)) with respect to the standard basis $\mathcal{S}$. Express A in terms of M and C. (You do not need to explicitly compute A.) Explain.

(d) Let F be the composition transformation defined by $F = R \circ T$, where R is the transformation which rotates a vector by an angle of $\pi/6$ radians around $\overrightarrow{e_1}$, in the direction from $\overrightarrow{e_2}$ toward $\overrightarrow{e_3}$. What is the matrix B for the transformation F with respect to the standard basis $\mathcal{S}$? (Express your answer in terms of M and C.)

3. (a) Compute the matrix product AB where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 6 & -2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 5 & 2 \\ 3 & -2 & 1 \\ 4 & 3 & 1 \end{pmatrix}$$

(b) Let A be given by the 2×3 matrix below, and let B be the $3 \times n$ matrix with rows $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \qquad B = \left(\begin{bmatrix} \text{———} \overrightarrow{v_1} \text{———} \\ \text{———} \overrightarrow{v_2} \text{———} \\ \text{———} \overrightarrow{v_3} \text{———} \end{bmatrix} \right)$$

Write the row vectors of the product AB as linear combinations of the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$.

4. Let the linear transformations below have matrices A, B, L, M, with domains and ranges as described in the diagram below:

$$\mathbb{R}^4 \xrightarrow{A} \mathbb{R}^2 \xrightarrow{B} \mathbb{R}^3 \xrightarrow{L} \mathbb{R}^1 \xrightarrow{M} \mathbb{R}^3$$

For the following problems, you might want to consider using the Rank-Nullity Theorem:

(a) Find the largest possible dimension for $C(A)$

(b) Find the smallest possible dimension for $N(BA)$

(c) Find the smallest possible dimension for $N(MLB)$

(d) Suppose that $N(B)$ has dimension 1; what are the possible dimensions of $C(MLBA)$? Explain.

5. Prove that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if and only if the determinant $ad - bc$ is not equal to zero.

Bonus Question– Prove or find a counterexample to the following statement:

Proposition: If an $n \times n$ matrix A has the property that

$$A^2 = 0_n$$

(where 0_n is the $n \times n$ matrix whose entries are all zero), then the matrix A must equal 0_n.