EXAM I SOLUTIONS

Math 51, Spring 2001.

You have 2 hours.

No notes, no books.

YOU MUST SHOW ALL WORK TO RECEIVE CREDIT

Good luck!

Name ______________________________

ID number____________________

“On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

Signature: ____________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

5. __________ (/20 points)

Bonus __________ (/10 points)

Total __________ (/100 points)

Circle your TA's name:

Kuan Ju Liu (2 and 6)

Robert Sussland (3 and 7)

Hunter Tart (4 and 8)

Alex Meadows (10)

Dana Rowland (11)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. (a) Define what it means for a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ to be "linearly dependent".

Solution: A collection of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is *linearly dependent* if any one of them can be expressed as a linear combination of the others.

or

A collection of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is *linearly dependent* if there exist constants $c_1, \ldots c_k$, not all of which are zero, such that

$$c_1 \vec{v}_1 + \ldots + c_k \vec{v}_k = 0$$

(b) Find a linear dependence of the three vectors below, or prove that they are independent.

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 7 \end{bmatrix}$$

Solution: We need to solve

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix that corresponds to this system is

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 2 & 5 & 0 \\ 4 & 1 & 7 & 0 \end{array}\right]$$

which row reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

There are no free variables, and we see that the only solution is the trivial solution. So, the three given vectors are independent.

(c) Prove that three nonzero vectors in $\mathbb{R}^2$ must be dependent.

Solution: Solving $a\overrightarrow{v}_1 + b\overrightarrow{v}_2 + c\overrightarrow{v}_3 = \overrightarrow{0}$ gives rise to a 2x3 matrix. Every 2x3 matrix has at most two pivots since there is at most one pivot per row; so there must be a column with no pivot, and thus there is a free variable. Choosing this free variable to be something other than zero and solving, we arrive at a nontrivial linear combination of the three vectors which is zero.

or

If the first two vectors are dependent, then we are done; if they are independent, then their span is all of R^2. In this case, the third vector must be a linear combination of the first two, and again we conclude that the vectors are dependent.

2. Suppose that V and W are vector subspaces of $\mathbb{R}^n$.

   (a) Show that the intersection $V \bigcap W$ (in other words, the set of all vectors that are in both V and W) is also a vector subspace of $\mathbb{R}^n$.

   Solution: We need to show that

   i. $\overrightarrow{x}, \overrightarrow{y} \in V \bigcap W \quad \Rightarrow \quad \overrightarrow{x+y} \in V \bigcap W$

   ii. $\overrightarrow{x} \in V \bigcap W \quad \Rightarrow \quad c\overrightarrow{x} \in V \bigcap W$

   Proofs:

   i. $\overrightarrow{x}, \overrightarrow{y} \in V \bigcap W \quad \Rightarrow \quad \overrightarrow{x}, \overrightarrow{y} \in V$ and $\overrightarrow{x}, \overrightarrow{y} \in W \quad \Rightarrow$

   $\overrightarrow{x+y} \in V$ and $\overrightarrow{x+y} \in W$ (since V, W are subspaces) $\quad \Rightarrow \quad \overrightarrow{x+y} \in V \bigcap W$

   ii. $\overrightarrow{x} \in V \bigcap W \quad \Rightarrow \quad \overrightarrow{x} \in V$ and $\overrightarrow{x} \in W \quad \Rightarrow$

   $c\overrightarrow{x} \in V$ and $c\overrightarrow{x} \in W$ (since V, W are subspaces) $\quad \Rightarrow \quad c\overrightarrow{x} \in V \bigcap W$

(b) Prove that the union $V \bigcup W$ (in other words, the set of all vectors that are in either V or W) is a vector space – or find a counterexample to refute this statement.

Solution: This statement is false. An easy counterexample is obtained with V and W being the $x-$ and $y-$ axes, in $\mathbb{R}^2$ respectively. Individually these are both vector subspaces of $\mathbb{R}^2$. However, their union is not.

In particular, their union does not satisfy the first condition:

$(1,0) \in V \bigcup W$ (since it is in V), and

$(0,1) \in V \bigcup W$ (since it is in W), but their sum

$(1,1)$ is not in $V \bigcup W$

Of course, there are many other counterexamples; for example, any two distinct lines through the origin in $\mathbb{R}^2$ can also serve as counterexamples.

3. (a) Find a parametric representation of the set of solutions to the system of equations below.

$$\begin{array}{rcrcrcr} 3x & + & 2y & + & z & = & -1 \\ -2x & & & + & 3z & = & 2 \end{array}$$

Solution: We row reduce the augmented matrix corresponding to the above system:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ -2 & 0 & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 0 & 4 & 11 & 4 \end{array}\right] \begin{array}{c} r_1 \\ 2r_1 + 3r_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & -9 & -6 \\ 0 & 4 & 11 & 4 \end{array}\right] \begin{array}{c} 2r_1 - r_2 \\ r_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3/2 & -1 \\ 0 & 1 & 11/4 & 1 \end{array}\right] \begin{array}{c} r_1/6 \\ r_2/4 \end{array}$$

So our solution vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ becomes $\begin{bmatrix} -1 + \frac{3}{2}z \\ 1 - \frac{11}{4}z \\ z \end{bmatrix}$,

which we rewrite as $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3/2 \\ -11/4 \\ 1 \end{bmatrix}$

This is our parametric solution.

(b) Given this result and your knowledge of the relationship between solution sets and the null space, find a basis for the null space of the matrix

$$\begin{pmatrix} 3 & 2 & 1 \\ -2 & 0 & 3 \end{pmatrix}$$

Solution: We know that the solution set to any system of linear equations can be written as the sum of a particular solution with the null space. So, we conclude from the above that the null space of the given matrix (which is the matrix of coefficients from the system above) is the second term of the expression above. So, the basis is given by the single direction vector,

$$\left\{ \begin{bmatrix} 3/2 \\ -11/4 \\ 1 \end{bmatrix} \right\}$$

(c) Use the cross product to find a basis for the null space of the matrix below (DO NOT row reduce as in parts (a) and (b); and make sure to **explain why your calculation is valid**.)

$$\begin{pmatrix} 2 & 4 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

Solution: We recall that the null space is precisely the set of vectors in the domain (in this case, $\mathbb{R}^3$) which are perpendicular to the row space; we can find such a vector by finding a vector which is perpendicular to the row vectors themselves.

Since there are two row vectors, and they are in $\mathbb{R}^3$, we can use the cross product (since the cross product of two vectors in $\mathbb{R}^3$ is perpendicular to both vectors).

$$\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (4)(2) - (-1)(0) \\ (1)(0) - (2)(2) \\ (2)(-1) - (1)(4) \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -6 \end{bmatrix}$$

This is therefore a vector in the null space.

Since the row space is clearly a plane through the origin in $\mathbb{R}^3$ (because the rows are independent), the null space must be a line through the origin. So, the vector obtained above is a basis for the null space.

4. (a) Suppose that the vectors $\overrightarrow{v}$, $\overrightarrow{w}$ and $\overrightarrow{x}$ are mutually perpendicular. Use dot products to find

$$\|\overrightarrow{v} + 3\overrightarrow{w} + 2\overrightarrow{x}\|$$

in terms of the lengths of $\overrightarrow{v}$, $\overrightarrow{w}$ and $\overrightarrow{x}$.

Solution: Since $\overrightarrow{v}$, $\overrightarrow{w}$ and $\overrightarrow{x}$ are mutually perpendicular, we know that their dot products are zero. So,

$$\|\overrightarrow{v} + 3\overrightarrow{w} + 2\overrightarrow{x}\|^2 = (\overrightarrow{v} + 3\overrightarrow{w} + 2\overrightarrow{x}) \cdot (\overrightarrow{v} + 3\overrightarrow{w} + 2\overrightarrow{x})$$

$$= (\overrightarrow{v} + 3\overrightarrow{w} + 2\overrightarrow{x}) \cdot \overrightarrow{v} + (\overrightarrow{v} + 3\overrightarrow{w} + 2\overrightarrow{x}) \cdot 3\overrightarrow{w} + (\overrightarrow{v} + 3\overrightarrow{w} + 2\overrightarrow{x}) \cdot 2\overrightarrow{x}$$

$$(\|\overrightarrow{v}\|^2 + 0 + 0) + (0 + 9\|\overrightarrow{w}\|^2 + 0) + (0 + 0 + 4\|\overrightarrow{x}\|^2)$$

$$= \|\overrightarrow{v}\|^2 + 9\|\overrightarrow{w}\|^2 + 4\|\overrightarrow{x}\|^2$$

Thus

$$\|\overrightarrow{v} + 3\overrightarrow{w} + 2\overrightarrow{x}\| = \sqrt{\|\overrightarrow{v}\|^2 + 9\|\overrightarrow{w}\|^2 + 4\|\overrightarrow{x}\|^2}$$

(b) Show that if a vector $\overrightarrow{v}$ is perpendicular to the row vectors $\overrightarrow{r_1}, \ldots \overrightarrow{r_m}$ of a matrix A, then the vector $\overrightarrow{v}$ must be in the null space of A.

Solution: If $\overrightarrow{v}$ is perpendicular to $\overrightarrow{r_1}, \ldots \overrightarrow{r_m}$, then we have that $\overrightarrow{v} \cdot \overrightarrow{r_1} = 0$, $\overrightarrow{v} \cdot \overrightarrow{r_2} = 0$,..., $\overrightarrow{v} \cdot \overrightarrow{r_m} = 0$.

So, we conclude that

$$A\overrightarrow{v} = \begin{bmatrix} \overrightarrow{v} \cdot \overrightarrow{r_1} \\ \overrightarrow{v} \cdot \overrightarrow{r_2} \\ \vdots \\ \overrightarrow{v} \cdot \overrightarrow{r_m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \overrightarrow{0}$$

Therefore $\overrightarrow{v}$ is in the null space of A.

5. The matrix A below has the given reduced row echelon form (You do not need to verify this).

$$A = \begin{pmatrix} 3 & 4 & 0 & 7 \\ 1 & -5 & 2 & -2 \\ -1 & 4 & 0 & 3 \\ 1 & -1 & 2 & 2 \end{pmatrix} \qquad \text{rref}(A) = \begin{pmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Using this information, write down bases for the row space, column space, and null space of A.

Solution: The pivots are boxed in the rref(A) matrix above.

(a) We showed in class that the basis for the row space is given by the nonzero rows of the reduced row echelon form of the matrix. So,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

are a basis for $R(A)$.

(b) We also showed in class that a basis for the column space is given by the columns of A that correspond to columns of rref(A) containing pivots. So,

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} \right\}$$

are a basis for $C(A)$.

(c) To find the basis for the null space, we first merely find the null space, by requiring that our vector satisfy the equations corresponding to the rref(A) matrix.

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -x_4 \\ -x_4 \\ -x_4 \\ x_4 \end{bmatrix} \right\} = \left\{ x_4 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

As we showed in class, the direction vector(s) for the parametric representation of the null space also form a basis for the null space. So, the basis is just

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Bonus Question: Suppose we have three matrices A, B_1, and B_2, with the following properties:

1. A is 4x4, B_1 is 4x3, and B_2 is 3x4
2. $A\overrightarrow{v} = B_1(B_2\overrightarrow{v})$ for all vectors $\overrightarrow{v}$ in $\mathbb{R}^4$

Show that there must exist a non-zero vector in $N(A)$, and that there must also exist a vector in $\mathbb{R}^4$ which is not in $C(A)$.

Solution:

1. First we observe that if $B_2\overrightarrow{v} = 0$, then by property (2), we must have $A\overrightarrow{v} = 0$. So, $N(B_2) \subset N(A)$.

   Since B_2 is a 3x4 matrix, there can be at most three pivots, and so there is at least one free variable. So, we conclude that $B_2\overrightarrow{v} = 0$ must have a nontrivial null space. And since $N(B_2) \subset N(A)$, then A must have a nontrivial null space also.

2. Property (2) also tells us that any linear combination of the columns of A is also a linear combination of the columns of B_1 (with coefficients given by the components of the vector $B_2\overrightarrow{v}$). So, we conclude that $C(A) \subset C(B_1)$.

   Since B_1 is a 4x3 matrix, there can be at most three pivots, and so there must be a row with no pivots (which must therefore be a row of zeroes). So, there must be some $\overrightarrow{b}$ such that $B_1\overrightarrow{x} = \overrightarrow{b}$ does not have a solution. Therefore, this $\overrightarrow{b}$ is not in $C(B_1)$, and so, since $C(A) \subset C(B_1)$, we conclude that $\overrightarrow{b}$ is not in $C(A)$ either.