

MATH 51 MIDTERM 2 SOLUTIONS (AUTUMN 2001)

1. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & -1 & 5 & 4 \\ 3 & 6 & -1 & 8 & 5 \\ 4 & 8 & -1 & 12 & 8 \end{bmatrix}.$$

Then

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(You do not need to verify this.)

(a) (3 points) Find a basis for $C(A)$.

Solution. The first, third, and fourth columns of $\text{rref}(A)$ have pivots, so

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 8 \\ 12 \end{bmatrix} \right\}$$

is a basis for $C(A)$.

(b) (4 points) Express each column of A which is *not* in your basis for $C(A)$ as a linear combination of your basis columns.

Solution. Denote the columns of A by \mathbf{v}_1 through \mathbf{v}_5 . Then $\mathbf{v}_1, \mathbf{v}_3$ and \mathbf{v}_4 are the basis vectors from part (a). By inspection $\mathbf{v}_2 = 2\mathbf{v}_1$. Looking at the columns \mathbf{w}_1 through \mathbf{w}_5 of $\text{rref}(A)$, it is clear that $\mathbf{w}_5 = -5\mathbf{w}_1 - 4\mathbf{w}_3 + 2\mathbf{w}_4$. Therefore, since the columns of A and $\text{rref}(A)$ share the same relations, it follows that $\mathbf{v}_5 = -5\mathbf{v}_1 - 4\mathbf{v}_3 + 2\mathbf{v}_4$.

(c) (3 points) What is the maximum number of linearly independent vectors which can be found in $N(A)$?

Solution. 2. The rank of A is 3, so the nullity of A is 2 by the Rank-Nullity Theorem.

2. (10 points) Answer each question True or False. No explanation is necessary.

- (a) If V is a linear subspace of \mathbf{R}^5 and $V \neq \mathbf{R}^5$ then any set of 5 vectors in V is linearly dependent.

Solution. True. (Any set of 5 linearly independent vectors in \mathbf{R}^5 must span \mathbf{R}^5 .)

- (b) If A is a 4×7 and if the dimension of $N(A)$ is 3 then for any \mathbf{b} in \mathbf{R}^4 the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Solution. True. (By the Rank-Nullity Theorem, $\text{rank}(A) = 4$, which implies that $C(A) = \mathbf{R}^4$. Therefore, $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b} \in \mathbf{R}^4$. Since the null space of A is nontrivial, the set of solutions for each \mathbf{b} is infinite.)

- (c) If $\mathbf{T} : \mathbf{R}^5 \rightarrow \mathbf{R}^3$ is an onto linear transformation and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors in \mathbf{R}^5 , then $\{\mathbf{T}(\mathbf{v}_1), \mathbf{T}(\mathbf{v}_2), \mathbf{T}(\mathbf{v}_3)\}$ spans \mathbf{R}^3 .

Solution. False. (For example, let \mathbf{T} be given by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then \mathbf{T} is onto since $\text{rank}(A) = 3$, and it is clear that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set. But

$$\mathbf{T}(\mathbf{v}_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{T}(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{T}(\mathbf{v}_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

clearly do not span \mathbf{R}^3 .)

- (d) If A is an 11×5 matrix and the linear transformation $\mathbf{T} : \mathbf{R}^5 \rightarrow \mathbf{R}^{11}$ defined by $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$ is one to one, then $\text{rank}(A) = 5$.

Solution. True. (Since \mathbf{T} is one to one, the nullity of A must be zero, so by the Rank-Nullity Theorem, $\text{rank}(A) = 5$.)

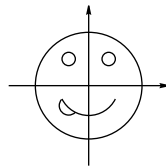
(e) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both one to one, then $g \circ f$ is one to one.

Solution. True. (Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then $g(f(x_1)) = g(f(x_2))$ and since g is one to one this implies $f(x_1) = f(x_2)$, and since f is one to one, this in turn implies $x_1 = x_2$.)

(f) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto, then $g \circ f$ is onto.

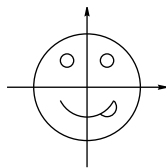
Solution. True. (Let $z \in Z$ be given. Then since g is onto, there is some element $y \in Y$ such that $g(y) = z$. Since f is onto, there is some element $x \in X$ such that $f(x) = y$. Therefore $(g \circ f)(x) = g(f(x)) = g(y) = z$, so $g \circ f$ is onto.)

3. (10 points) Let S denote the set of points in the face shown below.



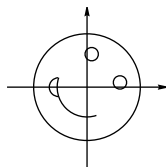
Each figure below is the image of S under the some transformation. Find the corresponding matrix.

(a)



Solution. Since $\mathbf{T}(\mathbf{e}_1) = -\mathbf{e}_1$ and $\mathbf{T}(\mathbf{e}_2) = \mathbf{e}_2$, the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

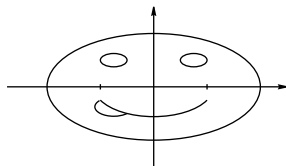
(b)



(Clockwise rotation by angle 45° .)

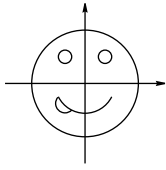
Solution. Since $\cos(-45^\circ) = \frac{1}{\sqrt{2}}$ and $\sin(-45^\circ) = -\frac{1}{\sqrt{2}}$, the matrix is $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

(c)



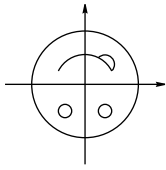
Solution. Since $\mathbf{T}(\mathbf{e}_1) = 2\mathbf{e}_1$ and $\mathbf{T}(\mathbf{e}_2) = \mathbf{e}_2$, the matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

(d)



The transformation is the identity, whose matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(e)



Solution. Since $\mathbf{T}(\mathbf{e}_1) = -\mathbf{e}_1$ and $\mathbf{T}(\mathbf{e}_2) = -\mathbf{e}_2$, the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

4. (a) (4 points) Suppose

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \end{bmatrix}.$$

Which of the products AB , BA , AC , CA make sense? For those that do, calculate them.

Solution. $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, BA is undefined, $AC = \begin{bmatrix} 9 & 12 & 0 \\ 19 & 26 & 0 \\ 29 & 40 & 0 \end{bmatrix}$ and $CA = \begin{bmatrix} 7 & 10 \\ 19 & 28 \end{bmatrix}$.

(b) (3 points) Suppose $D = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$. What is the area of the image of the unit circle under the transformation associated to D ?

Solution. The area of the unit circle is π and $\det(D) = -7$, so the image of the unit circle has area 7π .

(c) (3 points) Suppose E , F , and G are invertible matrices such that $EFGE^{-1} = GE$. Find F in terms of E , G , E^{-1} and G^{-1} .

Solution. First multiply both sides by E^{-1} on the left to get $FGE^{-1} = E^{-1}GE$. Next multiply both sides by E on the right to get $FG = E^{-1}GE^2$. Finally multiply both sides by G^{-1} on the right to get $F = E^{-1}GE^2G^{-1}$.

5. (a) (5 points) Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ -1 & -1 & 0 & 5 \\ 2 & 6 & 3 & -1 \\ 3 & 6 & 0 & 5 \end{bmatrix}$$

Solution. $\det(A) = -12$

(b) (5 points) Find the inverse of

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution. $B^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

6. (10 points) In each part below, state which figure represents the parametric curve defined by the given formula.

Figure 1

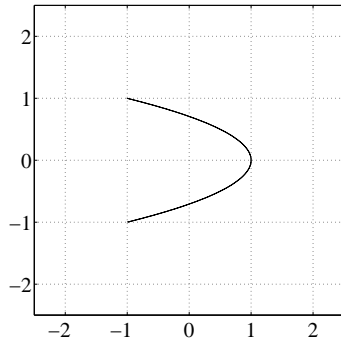


Figure 2

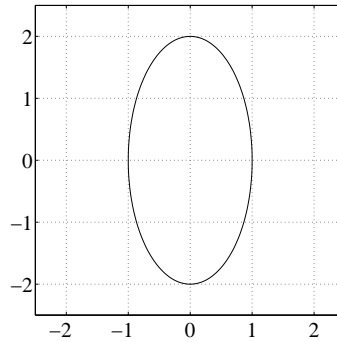


Figure 3

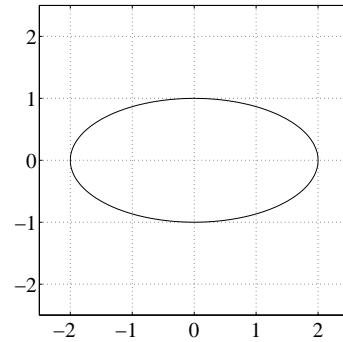


Figure 4

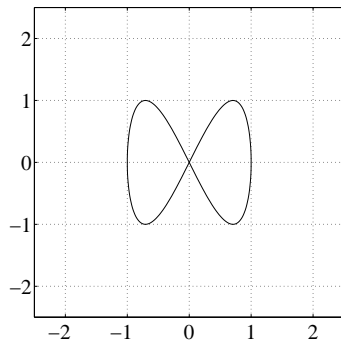


Figure 5

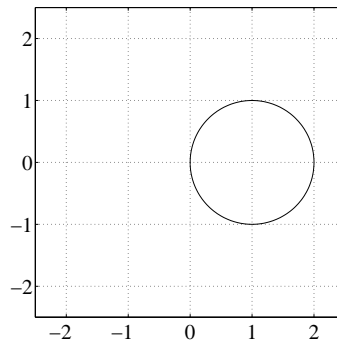
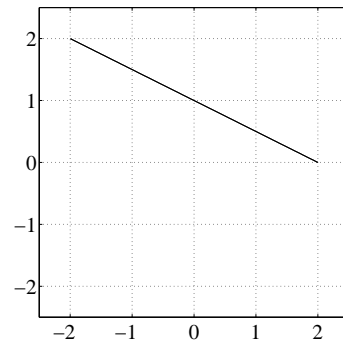


Figure 6



(a) $\mathbf{x}(t) = (1 + \cos t, \sin t)$

Solution. Figure 5.

(b) $\mathbf{x}(t) = (2 \cos t, \sin t)$

Solution. Figure 3.

(c) $\mathbf{x}(t) = (2 \cos t, 1 - \cos t)$

Solution. Figure 6.

(d) $\mathbf{x}(t) = (\cos(2t), \sin t)$

Solution. Figure 1.

(e) $\mathbf{x}(t) = (\cos t, \sin(2t))$

Solution. Figure 4.

7. A rocket begins with position $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and travels with velocity at time t given by

$$\mathbf{v}(t) = \begin{bmatrix} \cos t + \sin t \\ \cos t - \sin t \\ 1 \end{bmatrix}.$$

(a) (2 points) Find the acceleration of the rocket at time $t = \pi$.

Solution. $\mathbf{a}(t) = \mathbf{v}'(t) = \begin{bmatrix} -\sin t + \cos t \\ -\sin t - \cos t \\ 0 \end{bmatrix}$, so $\mathbf{a}(\pi) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

(b) (3 points) Find the position of the rocket at time $t = \pi$.

Solution.

$$\mathbf{x}(\pi) = \mathbf{x}(0) + \int_0^\pi \mathbf{v}(t) dt = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left. \begin{bmatrix} \sin t - \cos t \\ \sin t + \cos t \\ t \end{bmatrix} \right|_0^\pi = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ \pi \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ \pi \end{bmatrix}$$

(c) (2 points) Find the speed of the rocket at time $t = \pi$.

Solution. $s(\pi) = \|\mathbf{v}(\pi)\| = \left\| \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\| = \sqrt{3}$

(d) (3 points) Find a parametrization of the tangent line to the rocket's path at time $t = \pi$.

Solution. Using the values of $\mathbf{x}(\pi)$ and $\mathbf{v}(\pi)$ from parts (b) and (c), we have

$$\begin{bmatrix} 3 \\ -2 \\ \pi \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

8. In each part, state whether or not the indicated limit exists. If it exists, evaluate the limit (you do not need to prove your answer). If it does not exist, explain why.

(a) (3 points) $\lim_{(x,y) \rightarrow (1,2)} \frac{x^3}{3x+y}$.

Solution. $\frac{1}{5}$ (The function is rational and the denominator is nonzero at the point $(1, 2)$, so just plug in.)

(b) (3 points) $\lim_{(x,y) \rightarrow (0,0)} \frac{|x+y|}{|x|+|y|}$.

Solution. The limit does not exist. Along the line $x = 0$ the limit is 1, and along the line $y = -x$ the limit is zero.

(c) (4 points) $\lim_{(x,y) \rightarrow (1,1)} e^{x+y} \cdot \frac{x-y}{e^{x-y} - e^{-(x-y)}}$.

Solution. The term e^{x+y} goes to e^2 . Making the substitution $u = x - y$, the rest becomes

$$\lim_{u \rightarrow 0} \frac{u}{e^u - e^{-u}} = \lim_{u \rightarrow 0} \frac{1}{e^u + e^{-u}} = \frac{1}{2}$$

(using L'Hopital's Rule). Thus the limit equals $\frac{1}{2}e^2$.

9. (a) (4 points) Let $f(x, y) = x \sin y + e^{xy}$. Compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$.

Solution. $\frac{\partial f}{\partial x} = \sin y + ye^{xy}$, $\frac{\partial f}{\partial y} = x \cos y + xe^{xy}$, and $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos y + e^{xy}(1 + xy)$.

- (b) (6 points) Find an equation of the form $z = ax + by + c$ which describes the tangent plane to the graph of the function $g(x, y) = \sqrt{32 - 3x^2 - 5y^2}$ at the point $(1, 2, 3)$ on the graph of g .

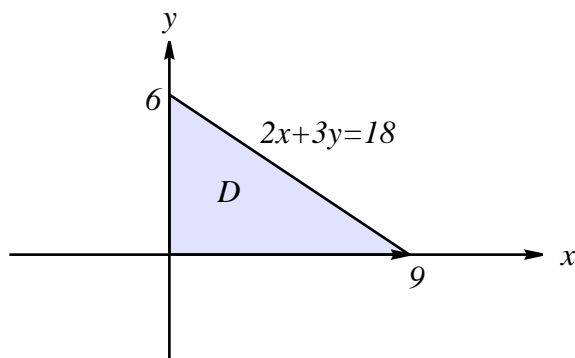
Solution. Since

$$\frac{\partial g}{\partial x}(x, y) = \frac{-3x}{\sqrt{32 - 3x^2 - 5y^2}} \quad \text{and} \quad \frac{\partial g}{\partial y}(x, y) = \frac{-5y}{\sqrt{32 - 3x^2 - 5y^2}},$$

we have $\frac{\partial g}{\partial x}(1, 2) = -1$ and $\frac{\partial g}{\partial y}(1, 2) = -\frac{10}{3}$ and therefore the equation of the tangent plane is $z = 3 - 1(x - 1) - \frac{10}{3}(y - 2)$, or equivalently $z = -x - \frac{10}{3}y + \frac{32}{3}$.

10. (a) (3 points) Sketch the set $D = \{(x, y) \in \mathbf{R}^2 \mid x \geq 0, y \geq 0 \text{ and } 18 - 2x - 3y \geq 0\}$.

Solution.



- (b) (7 points) Find the maximum and minimum of $f(x, y) = xy(18 - 2x - 3y)$ on the domain D from part (a).

Solution. First notice that on the boundary $x = 0$, $y = 0$, or $2x + 3y = 18$, $f(x, y) = 0$. Next, the first-order partial derivatives of f are $f_x = 18y - 4xy - 3y^2$ and $f_y = 18x - 2x^2 - 6xy$. Thus the critical points of f must satisfy

$$y(18 - 4x - 3y) = 0$$

$$x(18 - 2x - 6y) = 0$$

We have already considered all points which satisfy $x = 0$ or $y = 0$, so all that is left to consider are points that satisfy the linear system

$$4x + 3y = 18$$

$$2x + 6y = 18$$

The unique solution of this system is $(x, y) = (3, 2)$, so this is the only critical point of f in the interior of the domain D . Since $f(3, 2) = 36$, the maximum of f is 36 and the minimum of f is 0.