MATH 51 MIDTERM 1 (AUTUMN 2001)

Definition of Cross Product: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$

1. Suppose $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, θ is the angle between them, and $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

   Compute:

   (a) (2 points) $2\mathbf{u} - 3\mathbf{v}$,

   (b) (2 points) $\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$,

   (c) (2 points) $\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$,

   (d) (2 points) $\|\mathbf{u}\|$,

   (e) (2 points) $A\mathbf{u}$.

2. (10 points) Answer the following questions *true* or *false*. No explanation is required. You should be able to answer all of these questions without doing any calculations.

   (a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3.4 \\ 9 \\ 10 \end{bmatrix}, \begin{bmatrix} \pi \\ 14 \\ -1 \\ 1.2 \end{bmatrix}, \begin{bmatrix} 1.3 \\ -4 \\ 3 \\ -94 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ 10 \\ -764 \end{bmatrix} \right\}$ is a linearly independent set.

   (b) $\left\{ \begin{bmatrix} \pi \\ e \\ 7/3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2\pi \\ -2e \\ -14/3 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right\}$ is a linearly independent set.

   (c) $\left\{ \begin{bmatrix} \pi \\ e \\ 7/3 \\ 4 \end{bmatrix}, \begin{bmatrix} 11 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right\}$ spans $\mathbf{R}^4$.

   (d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 2 \end{bmatrix} \right\}$ spans $\mathbf{R}^3$.

   (e) If A is an $m \times n$ matrix and $\mathbf{b} \in \mathbf{R}^m$ is an arbitrary vector, then the set of solutions to $A\mathbf{x} = \mathbf{b}$ is always a linear subspace of $\mathbf{R}^n$.

   (f) A set of four vectors which span $\mathbf{R}^4$ must be linearly independent.

3. (a) (4 points) Find an equation for the plane going through the three points $A = (2, 1, 3)$, $B = (4, 1, 5)$, $C = (2, 3, 5)$ in $\mathbf{R}^3$. (Your answer should be of the form "$ax + by + cz = d$".)

(b) (3 points) Find the area of the triangle ABC.

(c) (3 points) Determine the angle between the two sides AB and AC.

4. (10 points) Is the set of vectors

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

linearly dependent, or linearly independent? If dependent, express one of the vectors as a linear combination of the others. If independent, explain why.

5. Let $\mathbf{v}$ and $\mathbf{w}$ be orthogonal vectors in $\mathbf{R}^n$ such that $\|\mathbf{v}\| = 2$ and $\|\mathbf{w}\| = 3$. Let $\mathbf{x} = 3\mathbf{v} + 4\mathbf{w}$ and $\mathbf{y} = 3\mathbf{v} - \mathbf{w}$.

   (a) (5 points) Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

   (b) (5 points) Compute $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$.

6. (10 points) Show that for every linearly independent set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ of vectors in $\mathbf{R}^n$, the set $\{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$ is also linearly independent.

7. (a) (5 points) Let

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 3 \\ 1 & 4 & 6 \\ 1 & 2 & 4 \end{bmatrix}.$$

   Find equations which must be satisfied by the components of $\mathbf{b}$ so that $A\mathbf{x} = \mathbf{b}$ has at least one solution. (Your answer should be one or more equations of the form $?\,b_1 + ?\,b_2 + ?\,b_3 + ?\,b_4 = ?$.)

   (b) (5 points) Suppose

$$\text{rref}(B) = \begin{bmatrix} 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & \pi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Find a set of vectors which spans the null space $N(B)$.

8. Consider the system of equations

$$\begin{array}{rcrcrcr} & & y & + & 2z & = & 3 \\ x & + & 2y & + & 3z & = & 6 \\ x & - & y & - & 3z & = & -3 \end{array}$$

   (a) (2 points) What is the corresponding augmented matrix?

   (b) (4 points) Perform Gaussian elimination to put the augmented matrix into row reduced echelon form.

(c) (4 points) Describe the solution set parametrically. Is it a point, a line, a plane, or something else?

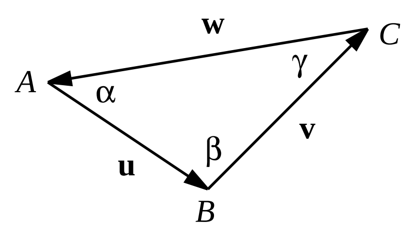
9. (a) (6 points) If V and W are linear subspaces of $\mathbf{R}^n$, define their sum $V + W$ by

$$V + W = \{\mathbf{v} + \mathbf{w} \mid \mathbf{v} \in V \text{ and } \mathbf{w} \in W\}.$$

Show that $V+W$ is also a linear subspace. That is, explain why each of the three subspace properties holds for $V + W$.

(b) (4 points) Show that the set $\{(x, y) \in \mathbf{R}^2 \mid x \text{ and } y \text{ are integers}\}$ is not a linear subspace of $\mathbf{R}^2$. Specify which properties fail, and why. (Recall that the set of integers is the set $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$ consisting of the positive and negative whole numbers and zero.)

10. **A proof of the Sine Law.** Consider the triangle ABC in $\mathbf{R}^3$. Let $\mathbf{u} = AB$, $\mathbf{v} = BC$, and $\mathbf{w} = CA$. Let $\alpha = \angle CAB$, $\beta = \angle ABC$, and $\gamma = \angle BCA$.



(a) (5 points) Show that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$. (*Hint:* Notice that $\mathbf{u}+\mathbf{v}+\mathbf{w} = \mathbf{0}$, and that $\mathbf{x} \times \mathbf{x} = \mathbf{0}$ for any vector $\mathbf{x}$ in $\mathbf{R}^3$.)

(b) (5 points) Use the result of part (a) to deduce the Sine Law:

$$\frac{|BC|}{\sin\alpha} = \frac{|CA|}{\sin\beta} = \frac{|AB|}{\sin\gamma}.$$