# MATH 51 FINAL EXAM SOLUTIONS (AUTUMN 2001)

1. Compute the following.

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$

**Solution.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

(b) The angle between $\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

**Solution.**

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-9}{3\sqrt{18}} = -\frac{\sqrt{2}}{2} \qquad \Longrightarrow \qquad \theta = \frac{3\pi}{4}$$

(c) The area of the triangle with vertices $(0,0,0)$, $(-1,4,1)$ and $(2,-2,1)$.

**Solution.** The area of this triangle is half the area of the parallelogram generated by $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$. Since $\mathbf{v}\times\mathbf{w} = \begin{bmatrix} 6 \\ 3 \\ -6 \end{bmatrix}$, the area of the triangle is $\frac{1}{2}\|\mathbf{v}\times\mathbf{w}\| = \frac{9}{2}$. Equivalently, using the result from part (b), the triangle has a base of $\|\mathbf{w}\| = 3$ and a height of $\|\mathbf{v}\|\sin\theta = 3$, so the area is $\frac{1}{2}\cdot 3\cdot 3 = \frac{9}{2}$.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 4 \\ 7 & 18 & 11 & 22 \end{bmatrix}.$$

(a) For which vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ does the equation $A\mathbf{x} = \mathbf{b}$ have a solution? Express your answer as one or more equations of the form $?b_1 + ?b_2 + ?b_3 = ?$.

**Solution.** Reducing the augmented matrix for the system $A\mathbf{x} = \mathbf{b}$ gives

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 1 & 3 & 2 & 4 & b_2 \\ 7 & 18 & 11 & 22 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 4 & 4 & 8 & b_3 - 7b_1 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 7b_1 - 4(b_2 - b_1) \end{array}\right]$$

The system is therefore consistent (i.e. $\mathbf{b}$ is in $C(A)$) if and only if $-3b_1 - 4b_2 + b_3 = 0$.

(b) Find a basis for the null space of $A$.

**Solution.** Continuing with the elimination from part (a) gives

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

so a basis for $N(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) Find a basis for the column space of $A$.

**Solution.** Since the pivots of rref$(A)$ are in the first two columns, the first two columns of $A$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 18 \end{bmatrix} \right\}$$

form a basis for $C(A)$.

(d) What is the rank of $A$?

**Solution.** 2

3. (a) Let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 1 \\ 4 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 3 \\ 4 \end{bmatrix}.$$

Express $\mathbf{b}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

**Solution.** Since

$$\operatorname{rref}\begin{bmatrix}1&3&1&1\\3&5&0&2\\0&2&4&3\\1&1&3&4\\2&4&4&5\end{bmatrix}=\begin{bmatrix}1&0&0&\frac{3}{2}\\0&1&0&-\frac{1}{2}\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

it follows that $\mathbf{b}=\frac{3}{2}\mathbf{v}_1-\frac{1}{2}\mathbf{v}_2+\mathbf{v}_3$.

(b) Assume $A\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=\begin{bmatrix}2\\0\\-1\end{bmatrix}$ and $\operatorname{rref}(A)=\begin{bmatrix}1&0&0&5\\0&0&1&-7\\0&0&0&0\end{bmatrix}$. Find all solutions of $A\mathbf{x}=\begin{bmatrix}2\\0\\-1\end{bmatrix}$.

**Solution.** From $\operatorname{rref}(A)$, we know that $N(A)=\operatorname{span}\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}-5\\0\\7\\1\end{bmatrix}\right)$. Thus the solutions are

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}+s\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-5\\0\\7\\1\end{bmatrix}$$

where $s$ and $t$ are any real numbers.

4. (a) Suppose $\mathbf{v}$ is a unit vector in $\mathbf{R}^n$. Show that, for any vector $\mathbf{w}\in\mathbf{R}^n$, the vector

$$\mathbf{w}-(\mathbf{w}\cdot\mathbf{v})\mathbf{v}$$

is orthogonal to $\mathbf{v}$.

**Solution.** Taking their dot product and using the fact that $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2=1$ gives

$$(\mathbf{w}-(\mathbf{w}\cdot\mathbf{v})\mathbf{v})\cdot\mathbf{v}=\mathbf{w}\cdot\mathbf{v}-(\mathbf{w}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{v})=\mathbf{w}\cdot\mathbf{v}-\mathbf{w}\cdot\mathbf{v}=0,$$

so they are orthogonal.

(b) Let $\mathbf{T}:\mathbf{R}^n\to\mathbf{R}^n$ be a linear transformation and let $V=\{\mathbf{x}\in\mathbf{R}^n \mid \mathbf{T}(\mathbf{x})=5\mathbf{x}\}$. Show that $V$ is a linear subspace of $\mathbf{R}^n$.

**Solution 1.** Verify the three subspace properties directly.

(i) $\mathbf{T}(\mathbf{0}) = \mathbf{0} = 5\mathbf{0}$, so $\mathbf{0}$ is in $V$.

(ii) Suppose $\mathbf{x}$ and $\mathbf{y}$ are in $V$. Then $\mathbf{T}(\mathbf{x}+\mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) = 5\mathbf{x} + 5\mathbf{y} = 5(\mathbf{x}+\mathbf{y})$, so $\mathbf{x}+\mathbf{y}$ is in $V$.

(iii) Suppose $\mathbf{x}$ is in $V$ and $c \in \mathbf{R}$. Then $\mathbf{T}(c\mathbf{x}) = c\mathbf{T}(\mathbf{x}) = c(5\mathbf{x}) = 5(c\mathbf{x})$, so $c\mathbf{x}$ is in $V$.

**Solution 2.** Let $A$ denote the matrix for $\mathbf{T}$. Then

$$V = \{\mathbf{x} \in \mathbf{R}^n \mid A\mathbf{x} = 5\mathbf{x}\} = \{\mathbf{x} \in \mathbf{R}^n \mid (A - 5I_n)\mathbf{x} = \mathbf{0}\} = N(A - 5I_n),$$

and the null space of any $n \times n$ matrix is a subspace of $\mathbf{R}^n$.

5. (a) Suppose $\mathbf{T} : \mathbf{R}^3 \to \mathbf{R}^5$ is a linear transformation such that

$$\mathbf{T}(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 3 \\ 4 \end{bmatrix} \qquad \mathbf{T}(\mathbf{e}_1 + \mathbf{e}_2) = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 5 \\ 3 \end{bmatrix} \qquad \mathbf{T}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}.$$

Find the matrix $A$ such that $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^3$.

**Solution.** The columns of $A$ are $\mathbf{T}(\mathbf{e}_1)$, $\mathbf{T}(\mathbf{e}_2)$ and $\mathbf{T}(\mathbf{e}_3)$. We are given $\mathbf{T}(\mathbf{e}_1)$,

$$\mathbf{T}(\mathbf{e}_2) = \mathbf{T}(\mathbf{e}_1 + \mathbf{e}_2) - \mathbf{T}(\mathbf{e}_1) = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \\ -1 \end{bmatrix},$$

and

$$\mathbf{T}(\mathbf{e}_3) = \mathbf{T}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) - \mathbf{T}(\mathbf{e}_1 + \mathbf{e}_2) = \begin{bmatrix} 3 \\ 2 \\ -2 \\ -1 \\ -2 \end{bmatrix}$$

so

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 2 \\ 5 & -1 & -2 \\ 3 & 2 & -1 \\ 4 & -1 & -2 \end{bmatrix}.$$

(b) The matrix for rotation by 45° about the $x$-axis in $\mathbf{R}^3$ is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and the matrix for rotation by 45° about the $z$-axis in $\mathbf{R}^3$ is

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(You need not verify these results.) Let $\mathbf{T}$ be the linear transformation obtained by first rotating by 45° about the $x$-axis and then rotating by 45° about the $z$-axis. Find the matrix for $\mathbf{T}$.

**Solution.** $BA = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

6. Consider the ellipse $2x^2 + 2xy + y^2 = 1$, and let $\mathbf{T} : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation with matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.

(a) Show that points $(u, v) = \mathbf{T}(x, y)$ in the image of the ellipse under $\mathbf{T}$ lie on the circle $u^2 + v^2 = 5$.

**Solution.** $u = x + 2y$ and $v = 3x + y$, so

$$\begin{aligned} u^2 + v^2 &= (x + 2y)^2 + (3x + y)^2 \\ &= x^2 + 4xy + 4y^2 + 9x^2 + 6xy + y^2 \\ &= 10x^2 + 10xy + 5y^2 \\ &= 5(2x^2 + 2xy + y^2) \\ &= 5 \end{aligned}$$

(b) Use the result of part (a) to find the area enclosed by the ellipse.

**Solution.** Since $\det(A) = -5$, the area of the circle is 5 times the area of the ellipse. Since the area of the circle is $5\pi$, the area of the ellipse is $\pi$.

(c) Parametrize the ellipse. Hint: Parametrize the circle first and use $A^{-1}$.

**Solution.** The circle is parametrized by $(u, v) = (\sqrt{5}\cos t, \sqrt{5}\sin t)$ for $0 \leq t \leq 2\pi$. Since $A^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}$,

$$\begin{aligned}(x, y) &= \frac{1}{5}(-u + 2v, 3u - v) \\ &= \frac{1}{\sqrt{5}}(-\cos t + 2\sin t, 3\cos t - \sin t).\end{aligned}$$

7. In each part determine which figure below represents the level curves of the given function.

   (a) $f(x, y) = x^2 + 3xy + y^2$

   **Solution.** Figure 4.

   (b) $f(x, y) = e^{x+y}$

   **Solution.** Figure 5.

   (c) $f(x, y) = \dfrac{y}{4x^2 + 1}$

   **Solution.** Figure 1.

   (d) $f(x, y) = 4x^2 + 5xy + 4y^2$

   **Solution.** Figure 3.

   (e) $f(x, y) = x - y$

   **Solution.** Figure 9.

Figure 1

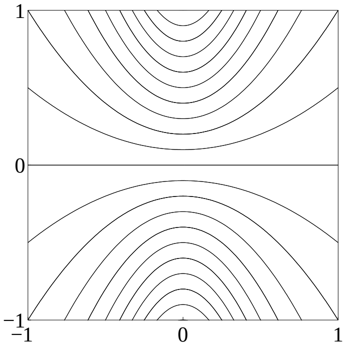


Figure 2

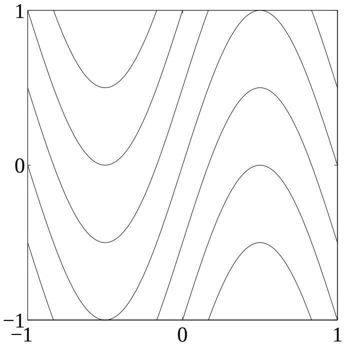


Figure 3

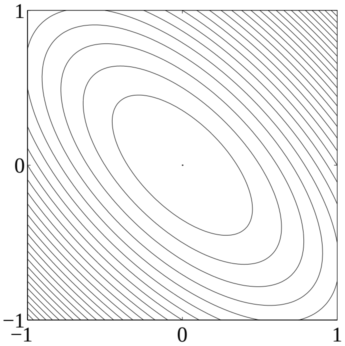


Figure 4

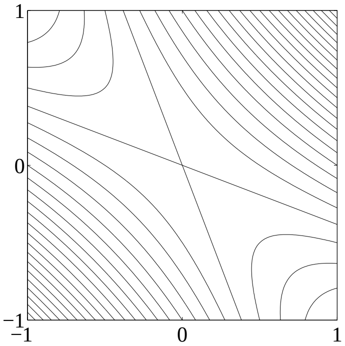


Figure 5

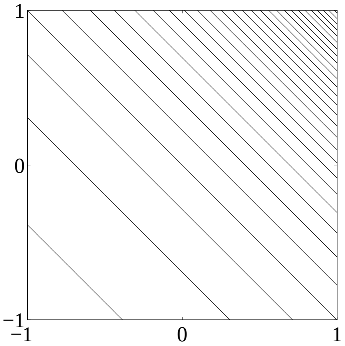


Figure 6

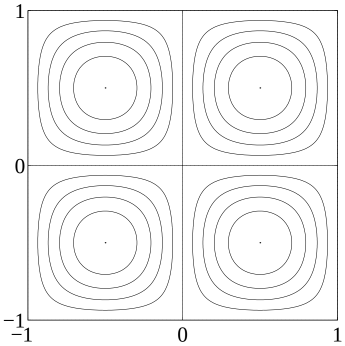


Figure 7

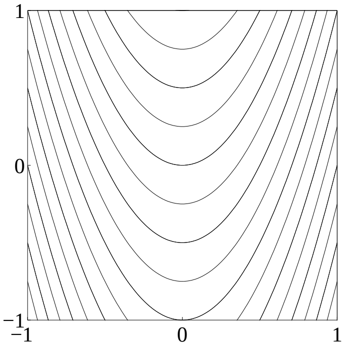


Figure 8

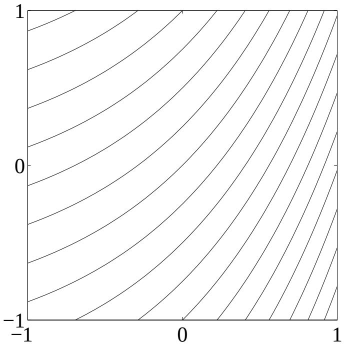
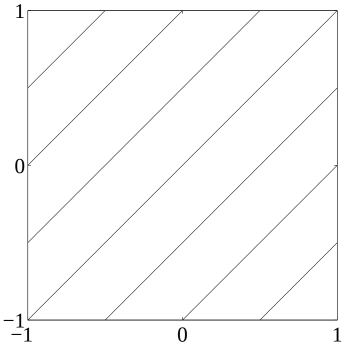


Figure 9



8. Answer each question True or False. No explanation is necessary. Each correct answer is worth 1 point.

   (a) There exists a number $c$ for which the function $g(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ c & (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$.

   **Solution.** False.

   (b) There exists a number $c$ for which the function $g(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ c & (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$.

**Solution.** True.

(c) On the domain $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ the function $f(x, y) = e^{x^2-2xy}\cos(xy)$ attains a maximum value.

**Solution.** True.

(d) On the domain $D = \{(x, y) \mid x^2 + y^2 < 1\}$ the function $f(x, y) = x + y$ attains a maximum value.

**Solution.** False.

(e) On the domain $D = \{(x, y) \mid x^2 + y^2 < 1\}$ the function $f(x, y) = 5$ attains a maximum value.

**Solution.** True.

(f) Suppose $f(x, y)$ is differentiable and $\nabla f(1, 2) = (3, -7)$. Then there exists a direction $\mathbf{u}$ in which $D_{\mathbf{u}} f(1, 2) = 8$.

**Solution.** False.

(g) If $f$ is differentiable at $\mathbf{a}$, then $D_{-\mathbf{u}} f(\mathbf{a}) = -D_{\mathbf{u}} f(\mathbf{a})$ for every unit vector $\mathbf{u}$.

**Solution.** True.

(h) If $f(x, y)$ has a local minimum at $(0, 0)$ along every line through $(0, 0)$, then $f$ has a local minimum at $(0, 0)$.

**Solution.** False.

(i) There exists a function $f(x, y)$ such that $\nabla f(x, y) = (2xy, x^2)$.

**Solution.** True.

(j) There exists a function $f(x, y)$ such that $\nabla f(x, y) = (x^2, 2xy)$.

**Solution.** False.

9. Find the maximum and minimum values of $f(x, y) = x^3 + 3x^2 - 9x + y^2 - 2y$ on the square domain $D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$ and all points at which they are attained.

**Solution.** $\nabla f(x,y) = (3x^2 + 6x - 9, 2y - 2) = (3(x-1)(x+3), 2(y-1))$, so the critical points of $f$ are $(1,1)$ and $(-3,1)$, but $(-3,1)$ is not in $D$.
On the boundary we must consider the vertices $(0,0)$, $(2,0)$, $(0,2)$, $(2,2)$, and the critical points $(1,0)$, $(1,2)$, $(0,1)$ and $(2,1)$. Evaluating $f$ at all of these points, we have

$$\begin{aligned}
&f(1,1) = -6\\
&f(0,0) = 0, f(2,0) = 2, f(0,2) = 0, f(2,2) = 2\\
&f(1,0) = -5, f(1,2) = -5, f(0,1) = -1, f(2,1) = 1
\end{aligned}$$

Thus the maximum of 2 is attained at $(2,0)$ and $(2,2)$, while the minimum of $-6$ is attained at $(1,1)$.

10. Let $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^3$ be given by $\mathbf{f}(s,t) = (t^2, st, e^s)$ and suppose $\mathbf{g} : \mathbf{R}^3 \to \mathbf{R}^2$ is differentiable with Jacobian matrix

$$J\mathbf{g}(x,y,z) = \begin{bmatrix} x & y & z \\ z & y & x \end{bmatrix}.$$

(a) Compute $J\mathbf{f}(1,2)$.

**Solution.** $J\mathbf{f}(s,t) = \begin{bmatrix} 0 & 2t \\ t & s \\ e^s & 0 \end{bmatrix}$, so $J\mathbf{f}(1,2) = \begin{bmatrix} 0 & 4 \\ 2 & 1 \\ e & 0 \end{bmatrix}$.

(b) Compute $J(\mathbf{g} \circ \mathbf{f})(1,2)$.

**Solution.** By the Chain Rule, since $\mathbf{f}(1,2) = (4,2,e)$,

$$\begin{aligned}
J(\mathbf{g} \circ \mathbf{f})(1,2) &= J\mathbf{g}(\mathbf{f}(1,2))J\mathbf{f}(1,2)\\
&= J\mathbf{g}(4,2,e)J\mathbf{f}(1,2)\\
&= \begin{bmatrix} 4 & 2 & e \\ e & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 2 & 1 \\ e & 0 \end{bmatrix}\\
&= \begin{bmatrix} 4 + e^2 & 18 \\ 4 + 4e & 4e + 2 \end{bmatrix}
\end{aligned}$$

11. Consider the surface defined by the equation

$$x^3 + xyz + z^3 = 3.$$

(a) Find the equation of the tangent plane to the surface at the point $(1,1,1)$.

**Solution.** Let $f(x,y,z) = x^3+xyz+z^3$. Then $\nabla f(x,y,z) = (3x^2+yz, xz, xy+3z^2)$, so $\nabla(f(1,1,1) = (4,1,4)$ is a vector normal to the tangent plane to the level surface $f(x,y,z) = 3$ at $(1,1,1)$. Thus the equation of the tangent plane is $4(x-1)+1(y-1)+4(z-1) = 0$.

(b) Regarding $z = z(x,y)$ as a function of $x$ and $y$ near the point $(1,1,1)$, compute $\dfrac{\partial z}{\partial x}(1,1)$.

**Solution 1.** Differentiate with respect to $x$ to get

$$3x^2 + yz + xy\frac{\partial z}{\partial x} + 3z^2\frac{\partial z}{\partial x} = 0.$$

At $(x,y,z) = (1,1,1)$ this gives $\frac{\partial z}{\partial x} = -1$.

**Solution 2.** Rewrite the equation of the tangent plane from part (a) as $z = 1 - 1(x-1) - \frac{1}{4}(y-1)$ and recall that the equation of the tangent plane to the graph of $z = f(x,y)$ at $(a,b,f(a,b))$ is given by

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(y-b)$$

so $\frac{\partial z}{\partial x}$ is just the coefficient if the $(x-1)$ term, $-1$.

12. Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a differentiable function and suppose that

$$\frac{\partial f}{\partial x}(x_0,y_0,z_0) = 4 \qquad \frac{\partial f}{\partial y}(x_0,y_0,z_0) = 5 \qquad \frac{\partial f}{\partial z}(x_0,y_0,z_0) = 8$$

(a) Let $\mathbf{u}$ be the unit vector $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$. Compute $D_{\mathbf{u}}f(x_0,y_0,z_0)$.

**Solution.**

$$D_{\mathbf{u}}f(x_0,y_0,z_0) = \nabla f(x_0,y_0,z_0) \cdot \mathbf{u} = (4,5,8) \cdot (1/3, 2/3, 2/3) = 10.$$

(b) Find a vector which points in the direction in which $f$ is decreasing most rapidly at $(x_0,y_0,z_0)$.

**Solution.** Any positive scalar multiple of $-\nabla f(x_0,y_0,z_0) = (-4,-5,-8)$.

(c) Suppose we know that $f(x_0,y_0,z_0) = 5$. Determine the gradient of the function $g(x,y,z) = (f(x,y,z))^2$ at $(x_0,y_0,z_0)$.

**Solution.** $\nabla g(x_0,y_0,z_0) = 2f(x_0,y_0,z_0)\nabla f(x_0,y_0,z_0) = (40,50,80)$.

13. Let $f(x,y) = x^2 - x\ln y$.

(a) Find $Jf(2,1)$.

**Solution.** $Jf(x,y) = \begin{bmatrix} 2x - \ln y & -\frac{x}{y} \end{bmatrix}$, so $Jf(2,1) = \begin{bmatrix} 4 & -2 \end{bmatrix}$.

(b) Find the linear approximation of $f$ at $(2,1)$ and use it to approximate $f(1.99, 1.02)$.

**Solution.** $f(2,1) = 4$, so $f(x,y) \approx 4 + 4(x-2) - 2(y-1)$, and thus $f(1.99, 1.02) \approx 3.92$.

(c) Find $Hf(2,1)$.

**Solution.** $Hf(x,y) = \begin{bmatrix} 2 & -\frac{1}{y} \\ -\frac{1}{y} & \frac{x}{y^2} \end{bmatrix}$, so $Hf(2,1) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

(d) Find the second degree Taylor Polynomial of $f$ at $(2,1)$.

**Solution.** $4 + 4(x-2) - 2(y-1) + \frac{1}{2}\left[2(x-2)^2 - 2(x-2)(y-1) + 2(y-1)^2\right]$

(e) Near $(2,1)$ does the graph of $f$ lie above its tangent plane, below its tangent plane, or neither? Explain.

**Solution.** $Hf(2,1)$ is positive definite since $AC - B^2 > 0$ and $A > 0$. Thus the graph of $f$ lies above its tangent plane near $(2,1)$.

14. (a) Find all the critical points of the function $f(x,y) = 12xy - 2x^2 - 9y^4$.

**Solution.** $\nabla f(x,y) = (12y - 4x, 12x - 36y^3)$. At a critical point therefore $x = 3y$, and thus $36y(1-y^2) = 0$. The critical points are therefore $(0,0)$, $(3,1)$ and $(-3,-1)$.

(b) At each critical point, determine whether $f$ has a local maximum, local minimum, or saddle point.

**Solution.** $Hf(x,y) = \begin{bmatrix} -4 & 12 \\ 12 & -108y^2 \end{bmatrix}$.

At the first critical point $Hf(0,0) = \begin{bmatrix} -4 & 12 \\ 12 & 0 \end{bmatrix}$ is indefinite since $AC - B^2 = -144 < 0$ and therefore $f$ has a saddle at $(0,0)$.

At the other two critical points $Hf(3,1) = Hf(-3,-1) = \begin{bmatrix} -4 & 12 \\ 12 & -108 \end{bmatrix}$ is negative definite since $AC - B^2 = 432 - 144 > 0$ and $A < 0$. Thus $f$ has local maxima at $(3,1)$ and $(-3,-1)$.

15. (a) Find the point on the ellipse defined by

$$x^2 + xy + y^2 = 7$$

at which the function $f(x, y) = 4x + 5y$ is maximized.

**Solution.** Let $g(x, y) = x^2 + xy + y^2$. Then $\nabla f(x, y) = \lambda \nabla g(x, y)$ leads to

$$\begin{aligned} 4 &= \lambda(2x + y) \\ 5 &= \lambda(x + 2y) \end{aligned}$$

Therefore $2x + y = \frac{4}{5}(x + 2y)$ which implies $y = 2x$. Using the constraint this implies $7x^2 = 7$, so $x = \pm 1$. Therefore the candidates are $(1, 2)$ and $(-1, -2)$. Clearly the maximum is $f(1, 2) = 14$.

(b) Find the point on the ellipse defined by

$$2x^2 + xy + 2y^2 = 30$$

which is closest to the line $x = 20$.

**Solution.** Let $f(x, y) = 2x^2 + xy + 2y^2$. The closest point will be a point at which the tangent line to the ellipse is vertical, so $\nabla f(x, y)$ is horizontal. That is $\frac{\partial f}{\partial y}(x, y) = x + 4y = 0$, so $x = -4y$. Using the equation, this gives $30y^2 = 30$, so $y = \pm 1$. Thus the candidates are $(-4, 1)$ and $(4, -1)$, and clearly $(4, -1)$ is closer to the line $x = 20$.