

MATH 51 FINAL EXAM (AUTUMN 2001)

1. Compute the following.

(a)  $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$

(b) The angle between  $\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ .

(c) The area of the triangle with vertices  $(0, 0, 0)$ ,  $(-1, 4, 1)$  and  $(2, -2, 1)$ .

2. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 4 \\ 7 & 18 & 11 & 22 \end{bmatrix}.$$

(a) For which vectors  $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$  does the equation  $A\mathbf{x} = \mathbf{b}$  have a solution? Express your answer as one or more equations of the form  $?b_1 + ?b_2 + ?b_3 = ?$ .

(b) Find a basis for the null space of  $A$ .

(c) Find a basis for the column space of  $A$ .

(d) What is the rank of  $A$ ?

3. (a) Let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 1 \\ 4 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 3 \\ 4 \end{bmatrix}.$$

Express  $\mathbf{b}$  as a linear combination of  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_3$ .

(b) Assume  $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$  and  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . Find all solutions of

$$A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

4. (a) Suppose  $\mathbf{v}$  is a unit vector in  $\mathbf{R}^n$ . Show that, for any vector  $\mathbf{w} \in \mathbf{R}^n$ , the vector

$$\mathbf{w} - (\mathbf{w} \cdot \mathbf{v})\mathbf{v}$$

is orthogonal to  $\mathbf{v}$ .

- (b) Let  $\mathbf{T} : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a linear transformation and let  $V = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{T}(\mathbf{x}) = 5\mathbf{x}\}$ . Show that  $V$  is a linear subspace of  $\mathbf{R}^n$ .

5. (a) Suppose  $\mathbf{T} : \mathbf{R}^3 \rightarrow \mathbf{R}^5$  is a linear transformation such that

$$\mathbf{T}(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 3 \\ 4 \end{bmatrix} \quad \mathbf{T}(\mathbf{e}_1 + \mathbf{e}_2) = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 5 \\ 3 \end{bmatrix} \quad \mathbf{T}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}.$$

Find the matrix  $A$  such that  $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$  for all  $\mathbf{x} \in \mathbf{R}^3$ .

- (b) The matrix for rotation by  $45^\circ$  about the  $x$ -axis in  $\mathbf{R}^3$  is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and the matrix for rotation by  $45^\circ$  about the  $z$ -axis in  $\mathbf{R}^3$  is

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

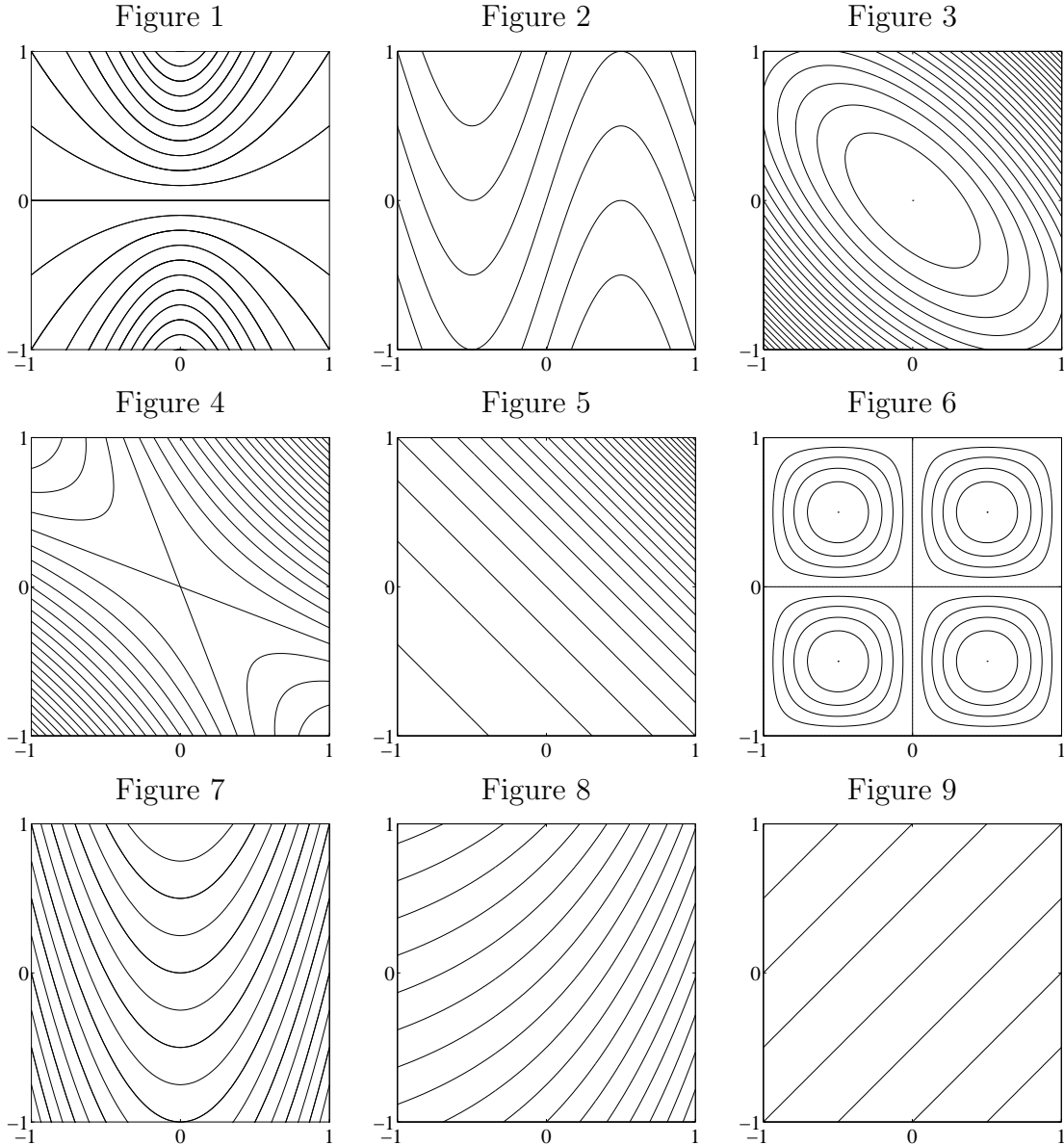
(You need not verify these results.) Let  $\mathbf{T}$  be the linear transformation obtained by first rotating by  $45^\circ$  about the  $x$ -axis and then rotating by  $45^\circ$  about the  $z$ -axis. Find the matrix for  $\mathbf{T}$ .

6. Consider the ellipse  $2x^2 + 2xy + y^2 = 1$ , and let  $\mathbf{T} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be the linear transformation with matrix  $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ .

- (a) Show that points  $(u, v) = \mathbf{T}(x, y)$  in the image of the ellipse under  $\mathbf{T}$  lie on the circle  $u^2 + v^2 = 5$ .  
 (b) Use the result of part (a) to find the area of the ellipse.  
 (c) Parametrize the ellipse. Hint: Parametrize the circle first and use  $A^{-1}$ .

7. In each part determine which figure below represents the level curves of the given function.

- (a)  $f(x, y) = x^2 + 3xy + y^2$   
 (b)  $f(x, y) = e^{x+y}$   
 (c)  $f(x, y) = \frac{y}{4x^2 + 1}$   
 (d)  $f(x, y) = 4x^2 + 5xy + 4y^2$   
 (e)  $f(x, y) = x - y$



8. Answer each question True or False. No explanation is necessary. Each correct answer is worth 1 point.

- (a) There exists a number  $c$  for which the function  $g(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ c & (x, y) = (0, 0) \end{cases}$  is continuous at  $(0, 0)$ .
- (b) There exists a number  $c$  for which the function  $g(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ c & (x, y) = (0, 0) \end{cases}$  is continuous at  $(0, 0)$ .
- (c) On the domain  $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$  the function  $f(x, y) = e^{x^2-2xy} \cos(xy)$  attains a maximum value.

- (d) On the domain  $D = \{(x, y) \mid x^2 + y^2 < 1\}$  the function  $f(x, y) = x + y$  attains a maximum value.
- (e) On the domain  $D = \{(x, y) \mid x^2 + y^2 < 1\}$  the function  $f(x, y) = 5$  attains a maximum value.
- (f) Suppose  $f(x, y)$  is differentiable and  $\nabla f(1, 2) = (3, -7)$ . Then there exists a direction  $\mathbf{u}$  in which  $D_{\mathbf{u}}f(1, 2) = 8$ .
- (g) If  $f$  is differentiable at  $\mathbf{a}$ , then  $D_{-\mathbf{u}}f(\mathbf{a}) = -D_{\mathbf{u}}f(\mathbf{a})$  for every unit vector  $\mathbf{u}$ .
- (h) If  $f(x, y)$  has a local minimum at  $(0, 0)$  along every line through  $(0, 0)$ , then  $f$  has a local minimum at  $(0, 0)$ .
- (i) There exists a function  $f(x, y)$  such that  $\nabla f(x, y) = (2xy, x^2)$ .
- (j) There exists a function  $f(x, y)$  such that  $\nabla f(x, y) = (x^2, 2xy)$ .
9. Find the maximum and minimum values of  $f(x, y) = x^3 + 3x^2 - 9x + y^2 - 2y$  on the square domain  $D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$  and all points at which they are attained.
10. Let  $\mathbf{f} : \mathbf{R}^2 \rightarrow \mathbf{R}^3$  be given by  $\mathbf{f}(s, t) = (t^2, st, e^s)$  and suppose  $\mathbf{g} : \mathbf{R}^3 \rightarrow \mathbf{R}^2$  is differentiable with Jacobian matrix

$$J\mathbf{g}(x, y, z) = \begin{bmatrix} x & y & z \\ z & y & x \end{bmatrix}.$$

- (a) Compute  $J\mathbf{f}(1, 2)$ .
- (b) Compute  $J(\mathbf{g} \circ \mathbf{f})(1, 2)$ .
11. Consider the surface defined by the equation

$$x^3 + xyz + z^3 = 3.$$

- (a) Find the equation of the tangent plane to the surface at the point  $(1, 1, 1)$ .
- (b) Regarding  $z = z(x, y)$  as a function of  $x$  and  $y$  near the point  $(1, 1, 1)$ , compute  $\frac{\partial z}{\partial x}(1, 1)$ .
12. Let  $f : \mathbf{R}^3 \rightarrow \mathbf{R}$  be a differentiable function and suppose that

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = 4 \quad \frac{\partial f}{\partial y}(x_0, y_0, z_0) = 5 \quad \frac{\partial f}{\partial z}(x_0, y_0, z_0) = 8$$

- (a) Let  $\mathbf{u}$  be the unit vector  $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ . Compute  $D_{\mathbf{u}}f(x_0, y_0, z_0)$ .
- (b) Find a vector which points in the direction in which  $f$  is decreasing most rapidly at  $(x_0, y_0, z_0)$ .

- (c) Suppose we know that  $f(x_0, y_0, z_0) = 5$ . Determine the gradient of the function  $g(x, y, z) = (f(x, y, z))^2$  at  $(x_0, y_0, z_0)$ .
13. Let  $f(x, y) = x^2 - x \ln y$ .
- (a) Find  $Jf(2, 1)$ .
  - (b) Find the linear approximation of  $f$  at  $(2, 1)$  and use it to approximate  $f(1.99, 1.02)$ .
  - (c) Find  $Hf(2, 1)$ .
  - (d) Find the second degree Taylor Polynomial of  $f$  at  $(2, 1)$ .
  - (e) Near  $(2, 1)$  does the graph of  $f$  lie above its tangent plane, below its tangent plane, or neither? Explain.
14. (a) Find all the critical points of the function  $f(x, y) = 12xy - 2x^2 - 9y^4$ .
- (b) At each critical point, determine whether  $f$  has a local maximum, local minimum, or saddle point.
15. (a) Find the point on the ellipse defined by

$$x^2 + xy + y^2 = 7$$

at which the function  $f(x, y) = 4x + 5y$  is maximized.

- (b) Find the point on the ellipse defined by

$$2x^2 + xy + 2y^2 = 30$$

which is closest to the line  $x = 20$ .