MATH 51 MIDTERM 2 SOLUTIONS

1. (a) Compute the inverse of the matrix

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -8 & 1 & -3/2 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -8 & 1 & -3/2 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3/2 & -8 \\ 0 & 1 & 0 & 0 & 1/2 & 2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

so

$$A^{-1} = \begin{bmatrix} 1 & -3/2 & -8 \\ 0 & 1/2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) For which value(s) of x is the matrix below **not** invertible? Explain your answer.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 5 & x & 6 \end{bmatrix}$$

Solution. The determinant of this matrix is

$$1(6-2x) - 1(0-10) + 1(0-5) = 11 - 2x$$

A matrix is not invertible if and only if its determinant equals zero, so this matrix is not invertible if and only if $x = 11/2$.

2. (a) Suppose

$$A = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}$$

is the matrix of a linear transformation which is geometrically a 60 degree rotation about a line L in $\mathbf{R}^3$. Find the matrix of a 120 degree rotation about L. Hint: Think about composition.

Solution. A 120 degree rotation is accomplished by composing the linear transformation with itself. Since the matrix for a composition of two transformations is the product of the matrices for the two transformations, the matrix for this rotation is

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) Let

$$B = \begin{bmatrix} 2 & 2 & 3 & 5 \\ 4 & 3 & 2 & 1 \\ -1 & 2 & -1 & 2 \\ 9 & 8 & 5 & 8 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 6 \\ 4 \\ -2 \\ 10 \end{bmatrix}$$

Compute $B^{-1}\mathbf{v}$. Hint: You do not need to compute B^{-1}. Compare $\mathbf{v}$ with the columns of B.

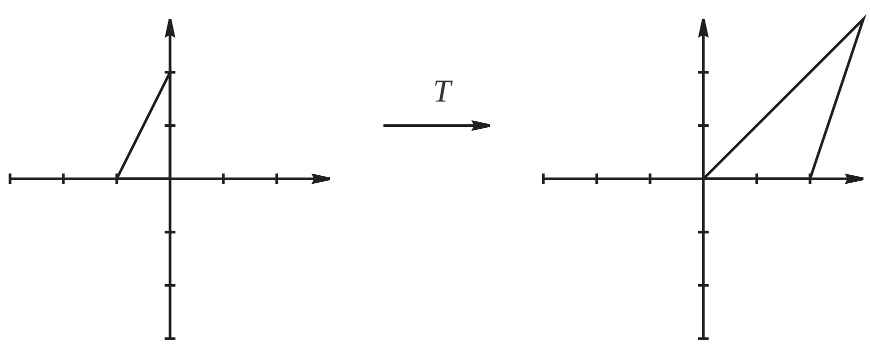
Solution. The vector $\mathbf{v}$ is twice the third column of B. Since $B\mathbf{e}_3$ equals the third column of B, multiplying by 2 gives

$$\mathbf{v} = 2B\mathbf{e}_3 = B(2\mathbf{e}_3)$$

and therefore

$$B^{-1}\mathbf{v} = 2\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

3. Let Δ_1 be the triangle with vertices $(0,0)$, $(-1,0)$ and $(0,2)$ and let Δ_2 be the triangle with vertices $(0,0)$, $(2,0)$ and $(3,3)$. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation such that $T(\Delta_1) = \Delta_2$.



(a) There are exactly two such linear transformations. Find the matrix for one of them.

Solution. Since $T(0,0) = (0,0)$, there are two possibilities. If $T(-1,0) = (2,0)$ and $T(0,2) = (3,3)$ we have

$$T(-\mathbf{e}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad T(\mathbf{e}_1) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$T(2\mathbf{e}_2) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$$

and thus

$$A = \begin{bmatrix} -2 & 3/2 \\ 0 & 3/2 \end{bmatrix}$$

If $T(-1,0) = (3,3)$ and $T(0,2) = (2,0)$ we have

$$T(-\mathbf{e}_1) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad T(\mathbf{e}_1) = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$T(2\mathbf{e}_2) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and thus

$$A = \begin{bmatrix} -3 & 1 \\ -3 & 0 \end{bmatrix}$$

(b) Let E represent the region bounded by the ellipse

$$\frac{x^2}{4} + \frac{y^2}{25} = 1$$

The area of E is 10π. Find the area of $T(E)$. Note: The answer is the same for both linear transformations T which satisfy $T(\Delta_1) = \Delta_2$.

Solution. The area of $T(E)$ is $|\det A|$ times the area of E. Since $|\det A| = 3$ (for either A above), the area of $T(E)$ is 30π.

4. Let

$$A = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 1 \\ 5 & -7 & 6 \end{bmatrix}$$

(a) Find the eigenvalues of A.

Solution. The characteristic polynomial is

$$\det(\lambda I_2 - A) = (\lambda - 5)(\lambda - 3) - 1 = \lambda^2 - 8\lambda + 14$$

The eigenvalues are its roots,

$$\lambda = \frac{8 \pm \sqrt{64 - 56}}{2} = 4 \pm \sqrt{2}$$

(b) $\lambda = 3$ is an eigenvalue of B. (You do not need to check this.) Find all eigenvectors of B with eigenvalue 3.

Solution. The eigenvectors with eigenvalue 3 are **nonzero** solutions of $B\mathbf{v} = 3\mathbf{v}$, or equivalently, nonzero solutions of $(3I_3 - B)\mathbf{v} = \mathbf{0}$. Since

$$3I_3 - B = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & -1 \\ -5 & 7 & -3 \end{bmatrix}$$

has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

the solutions of $(3I_3 - B)\mathbf{v} = \mathbf{0}$ are

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1/4v_3 \\ 1/4v_3 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -1/4 \\ 1/4 \\ 1 \end{bmatrix}$$

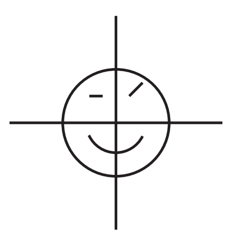
Thus the set of eigenvectors with eigenvalue 3 can be written as

$$\left\{ c \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} : c \neq 0 \right\}$$

5. Let

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

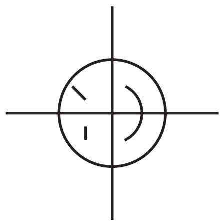
Let S denote the set of points in the face shown below.



Each figure below is the image of S under the linear transformation corresponding to one of the matrices above. Match each figure with the corresponding matrix.

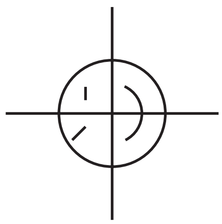
Solution. The matrix for a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ is determined by finding $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$. This can be done by observing where the right and top sides of the face are sent by the transformation.

(a)



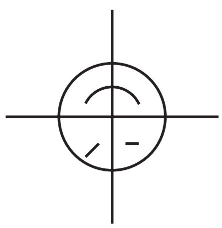
$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, so the answer is E.

(b)



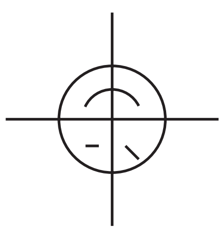
$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, so the answer is A.

(c)



$T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, so the answer is D.

(d)



$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, so the answer is C.

6. (a) Compute the following limit. Explain your answer.

$$\lim_{(x,y,z)\to(2,3,-1)} \frac{xy^2z - 2xyz}{x^2y + xz + y^2z^2}$$

Solution. Since $2^2(3) + 2(-1) + 3^2(-1)^2 = 19$, the denominator does not vanish at $(2, 3, -1)$. Since both the numerator and denominator are polynomials, which are continuous everywhere, and since the quotient of continuous functions is continuous whenever the denominator is nonzero, this function is continuous at $(2, 3, -1)$. The limit is therefore found by evalueting the function at $(2, 3, -1)$. Thus the limit equals

$$\frac{2(3^2)(-1) - 2(2)(3)(-1)}{2^2(3) + 2(-1) + 3^2(-1)^2} = -\frac{6}{19}$$

(b) Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 + y^2}{x^2 + 2y^2}$$

Solution. Along the line $y = 0$ we have

$$\lim_{x\to 0} \frac{2x^2}{x^2} = \lim_{x\to 0} 2 = 2$$

while along the line $x = 0$ we have

$$\lim_{y\to 0} \frac{y^2}{2y^2} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

Since the limit is different along these lines, the limit does not exist.

7. Let $f(x, y) = xy + \sin(2x - 4y)$.

(a) Suppose an ant is crawling on a surface whose height in cm at the point (x, y) is given by $f(x, y)$. If the ant is crawling in such a way that its x-coordinate is increasing at $2cm/sec$ and its y-coordinate is increasing at $1cm/sec$, at what rate is its height changing when the (x, y) coordinates of the ant are $(2, 1)$?

Solution. By the Chain Rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

We are given

$$\frac{dx}{dt} = 2 \qquad \text{and} \qquad \frac{dy}{dt} = 1$$

Since

$$\frac{\partial f}{\partial x}(x, y) = y + 2\cos(2x - 4y) \qquad \frac{\partial f}{\partial y}(x, y) = x - 4\cos(2x - 4y)$$

$$\frac{\partial f}{\partial x}(2, 1) = 3 \qquad \frac{\partial f}{\partial y}(2, 1) = -2$$

we have

$$\frac{df}{dt} = 3(2) + (-2)1 = 4$$

so the ant is ascending at $4cm/sec$.

(b) Find $\dfrac{\partial^2 f}{\partial y \partial x}(x, y)$ and $\dfrac{\partial^2 f}{\partial x^2}(x, y)$.

Solution.

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = 1 + 8\cos(2x - 4y) \qquad \frac{\partial^2 f}{\partial x^2}(x, y) = -4\sin(2x - 4y)$$

8. Let $f : D \subset \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x, y) = \sqrt{xy + y^2}$.

(a) Sketch the domain D of f. Hint: $xy + y^2 = y(x + y)$.

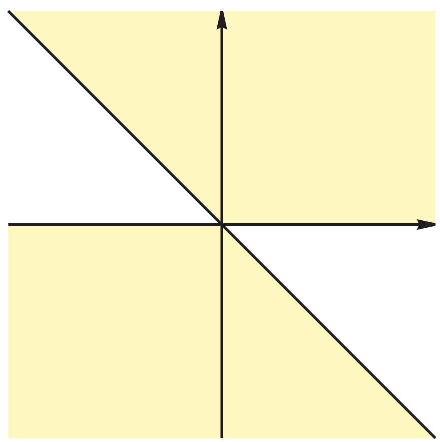
Solution. The expression under the radical must be nonnegative for $f(x, y)$ to be defined. Since $y(x + y) \geq 0$ implies

$$y \geq 0 \qquad \text{and} \qquad x + y \geq 0$$

or

$$y \leq 0 \qquad \text{and} \qquad x + y \leq 0$$

the domain D takes the form shown below.



(b) Find $Jf(3, 1)$.

Solution.

$$Jf(x, y) = \begin{bmatrix} \dfrac{y}{2\sqrt{xy + y^2}} & \dfrac{x + 2y}{2\sqrt{xy + y^2}} \end{bmatrix}$$

so

$$Jf(3, 1) = \begin{bmatrix} \dfrac{1}{4} & \dfrac{5}{4} \end{bmatrix}$$

(c) Use the answer to part (b) to find an approximation of $f(3.01, 1.02)$.

Solution.

$$f(3.01, 1.02) \approx f(3, 1) + Jf(3, 1)\begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix} = 2 + \frac{0.01}{4} + \frac{0.1}{4} = 2.0275$$

9. Define $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^3$ and $\mathbf{g} : \mathbf{R}^3 \to \mathbf{R}^2$ by

$$\begin{aligned} \mathbf{f}(x,y) &= (xy, x^2 + y^2, 2x - 2y) \\ \mathbf{g}(x,y,z) &= (x^2 + y^2 + z^2, xyz) \end{aligned}$$

Find the following Jacobian matrices.

(a) $J\mathbf{f}(1,1)$.

Solution.

$$J\mathbf{f}(x,y) = \begin{bmatrix} y & x \\ 2x & 2y \\ 2 & -2 \end{bmatrix}$$

so

$$J\mathbf{f}(1,1) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & -2 \end{bmatrix}$$

(b) $J\mathbf{g}(1,2,0)$.

Solution.

$$J\mathbf{g}(x,y,z) = \begin{bmatrix} 2x & 2y & 2z \\ yz & xz & xy \end{bmatrix}$$

so

$$J\mathbf{g}(1,2,0) = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c) $J(\mathbf{g} \circ \mathbf{f})(1,1)$.

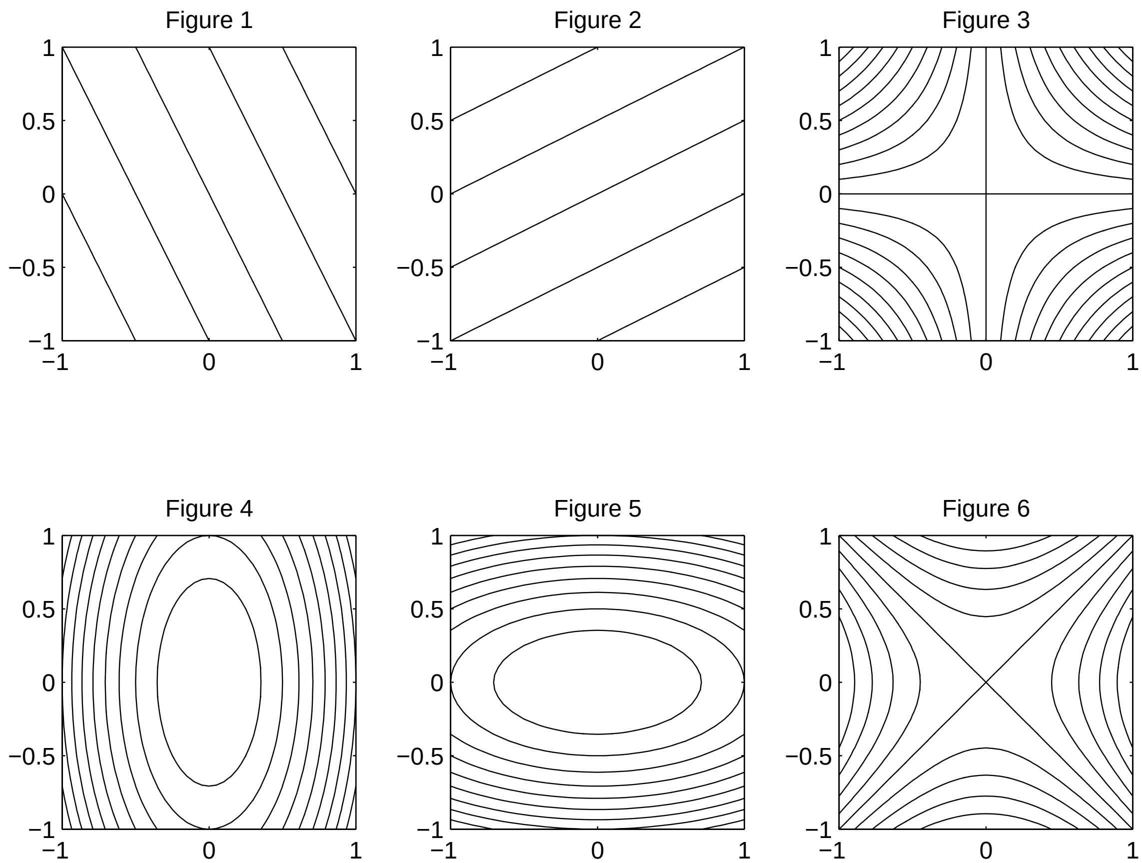
Solution. Since $\mathbf{f}(1,1) = (1,2,0)$,

$$J(\mathbf{g} \circ \mathbf{f})(1,1) = J\mathbf{g}(\mathbf{f}(1,1))J\mathbf{f}(1,1) = J\mathbf{g}(1,2,0)J\mathbf{f}(1,1)$$

Therefore, from parts (a) and (b) it follows that

$$J(\mathbf{g} \circ \mathbf{f})(1,1) = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 4 & -4 \end{bmatrix}$$

10. Each figure below represents the level curves of some function. (The graphs are shown in the usual orientation, with the x-axis horizontal and the y-axis vertical.)



For each function below, indicate which figure represents its level curves.

(a) $x - 2y$. **Solution.** Figure 2.

(b) xy. **Solution.** Figure 3.

(c) $x^2 + 4y^2$. **Solution.** Figure 5.