

so the system reduces to

$$\begin{aligned}x - z &= 2 \\ y + 2z &= 4\end{aligned}$$

Solving for the pivot variables x and y gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + z \\ 4 - 2z \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

3. (a) Suppose \mathbf{u} and \mathbf{v} are vectors in \mathbf{R}^n such that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal (i.e., perpendicular) to each other. Show that $\|\mathbf{u}\| = \|\mathbf{v}\|$.

Solution. Two vectors are orthogonal if and only if their dot product is zero. So

$$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$$

The left hand side expands to

$$\begin{aligned}\mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2\end{aligned}$$

Thus $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$, so $\|\mathbf{u}\| = \|\mathbf{v}\|$.

- (b) Suppose \mathbf{u} , \mathbf{v} , and \mathbf{w} are unit vectors in \mathbf{R}^n . (Recall that a unit vector is a vector whose length is 1.) Suppose each vector is orthogonal (i.e., perpendicular) to each of the other two. Show that the two vectors

$$\mathbf{u} - 3\mathbf{v} + 2\mathbf{w} \quad \text{and} \quad \mathbf{u} + \mathbf{v} + \mathbf{w}$$

are orthogonal to each other.

Solution. Since \mathbf{u} , \mathbf{v} and \mathbf{w} are unit vectors, their lengths (and hence their lengths squared) are all equal to 1. So $\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{w} = 1$. Since each vector is perpendicular to the others, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = 0$, $\mathbf{u} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0$ and $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = 0$. So the dot product of the two vectors given is

$$\begin{aligned}(\mathbf{u} - 3\mathbf{v} + 2\mathbf{w}) \cdot (\mathbf{u} + \mathbf{v} + \mathbf{w}) &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \\ &\quad - 3\mathbf{v} \cdot \mathbf{u} - 3\mathbf{v} \cdot \mathbf{v} - 3\mathbf{v} \cdot \mathbf{w} \\ &\quad + 2\mathbf{w} \cdot \mathbf{u} + 2\mathbf{w} \cdot \mathbf{v} + 2\mathbf{w} \cdot \mathbf{w} \\ &= 1 - 3 + 2 = 0\end{aligned}$$

so they are orthogonal.

4. Consider the points $A = (1, 1, 1)$, $B = (1, 3, 1)$ and $C = (1, 1, 4)$ in \mathbf{R}^3 .

- (a) Find the cosine of the angle at B of the triangle ABC .

Solution. The vector from B to A is

$$\mathbf{u} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

and the vector from B to C is

$$\mathbf{v} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

The angle at B is the angle θ between \mathbf{u} and \mathbf{v} . Since $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)$ it follows that

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{4}{2\sqrt{13}} = \boxed{\frac{2}{\sqrt{13}}}$$

(b) Find a parametric equation for the plane through the points A , B , and C .

Solution. The vectors \mathbf{u} and \mathbf{v} found in part (a) are parallel to the plane, and

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is in the plane, so

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

is a parametric representation of the plane. There are many others.

5. Are the following three vectors in \mathbf{R}^4 linearly independent or linearly dependent? Show your work and explain your answer.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 4 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 8 \\ 2 \end{bmatrix}$$

Solution. The vectors will be dependent if and only if there exist nonzero solutions c_1, c_2, c_3 of $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$. This vector equation is equivalent to the system

$$\begin{aligned} c_1 - c_2 + 3c_3 &= 0 \\ 2c_1 + 2c_2 + 3c_3 &= 0 \\ 3c_1 - 2c_2 + 8c_3 &= 0 \\ -c_1 + 4c_2 + 2c_3 &= 0 \end{aligned}$$

The coefficient matrix of this system is

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 2 & 2 \\ 3 & -2 & 8 \\ -1 & 4 & 2 \end{bmatrix}$$

The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the system reduces to

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

Thus only the trivial combination of \mathbf{u} , \mathbf{v} and \mathbf{w} equals the zero vector, so these vectors are linearly independent.

6. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 1 & 3 & 7 \\ 2 & 2 & 2 \end{bmatrix}.$$

What condition(s) must \mathbf{b} satisfy to be in the column space of A ? (Your answer should be one or more equations of the form $?b_1 + ?b_2 + ?b_3 + ?b_4 = ?$.)

Solution. The augmented matrix for the system $A\mathbf{x} = \mathbf{b}$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 1 & 3 & b_2 \\ 1 & 3 & 7 & b_3 \\ 2 & 2 & 2 & b_4 \end{array} \right]$$

Putting the left hand side of the matrix in reduced row echelon form, we get

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 1 & 3 & b_2 \\ 0 & 1 & 3 & b_3 - b_1 \\ 0 & -2 & -6 & b_4 - 2b_1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & b_1 - 2b_2 \\ 0 & 1 & 3 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \\ 0 & 0 & 0 & b_4 - 2b_1 + 2b_2 \end{array} \right]$$

For a solution to exist, the terms on the right in the last two rows must equal zero. So \mathbf{b} must satisfy

$$\boxed{\begin{array}{rcl} -b_1 & - & b_2 & + & b_3 & & = & 0 \\ -2b_1 & + & 2b_2 & & & + & b_4 & = & 0 \end{array}}$$

to be in the column space of A .

7. (a) Suppose \mathbf{u} is a vector in \mathbf{R}^4 . Let V be the set of all vectors in \mathbf{R}^4 which are orthogonal (i.e. perpendicular) to \mathbf{u} . That is,

$$V = \{\mathbf{x} \in \mathbf{R}^4 \mid \mathbf{x} \cdot \mathbf{u} = 0\}.$$

Show that V is a subspace of \mathbf{R}^4 .

Solution. Since $\mathbf{0} \cdot \mathbf{u} = 0$, $\mathbf{0}$ is in V , so V is nonempty. Now we need to verify the two subspace properties.

1. Let \mathbf{x}, \mathbf{y} be vectors in V . Then $\mathbf{x} \cdot \mathbf{u} = 0$ and $\mathbf{y} \cdot \mathbf{u} = 0$. So

$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{u} + \mathbf{y} \cdot \mathbf{u} = 0 + 0 = 0$$

which means $\mathbf{x} + \mathbf{y}$ is in V . So V is closed under addition.

2. Let \mathbf{x} be a vector in V and c any scalar. Since $\mathbf{x} \cdot \mathbf{u} = 0$,

$$(c\mathbf{x}) \cdot \mathbf{u} = c(\mathbf{x} \cdot \mathbf{u}) = c0 = 0$$

so $c\mathbf{x}$ is in V . Thus V is closed under scalar multiplication.

(b) Suppose the vector \mathbf{u} in part (a) is

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Find a basis for V .

Solution. A vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is in V if $\mathbf{x} \cdot \mathbf{u} = 0$. That is,

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

This system of one equation is in reduced row echelon form. Solving for the pivot variable x_1 we get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 - 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for V .

(c) What is the dimension of the subspace V in part (b)?

Solution. The basis for V found in part (b) had 3 elements, so the dimension of V is $\boxed{3}$.

8. Parts (a), (b) and (c) of this question all refer to the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 & -3 \\ 2 & 4 & 1 & -3 & 5 \\ 1 & 2 & 2 & 1 & 12 \\ -3 & -6 & -2 & -1 & -20 \end{bmatrix}$$

The reduced echelon form for A is (you do not need to check this)

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the column space $C(A)$ of A .

Solution. The pivots of R are in columns 1, 3, and 4. The corresponding column vectors

$$\left[\begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \end{bmatrix} \right]$$

of A form a basis for $C(A)$.

(b) Find a basis for the null space $N(A)$ of A .

Solution. The system $R\mathbf{x} = \mathbf{0}$ takes the form

$$\begin{array}{rclcl} x_1 & + & 2x_2 & & + & 4x_5 & = & 0 \\ & & & x_3 & & + & 3x_5 & = & 0 \\ & & & & x_4 & + & 2x_5 & = & 0 \end{array}$$

Solving for the pivot variables x_1, x_3, x_4 gives

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_5 \\ x_2 \\ -3x_5 \\ -2x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ -3 \\ -2 \\ 1 \end{bmatrix}$$

Since $N(A) = N(R)$, the vectors

$$\left[\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ -2 \\ 1 \end{bmatrix} \right]$$

form a basis for $N(A)$

(c) If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$ then $A\mathbf{v} = \begin{bmatrix} 9 \\ 1 \\ 4 \\ -12 \end{bmatrix}$. (You do not need to check this.)

Find all solutions of

$$A\mathbf{x} = \begin{bmatrix} 9 \\ 1 \\ 4 \\ -12 \end{bmatrix}.$$

Solution. Since the vector \mathbf{v} is one solution, every solution takes the form $\mathbf{v} + \mathbf{x}_n$ where \mathbf{x}_n is in $N(A)$. Thus the solutions are

$$\boxed{\begin{bmatrix} 2 \\ -1 \\ -4 \\ 0 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 0 \\ -3 \\ -2 \\ 1 \end{bmatrix}}$$

where c_1 and c_2 are arbitrary real numbers.

9. Let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 2 & -4 \end{bmatrix}$$

Compute each the following, if possible.

(a) $3\mathbf{x} + 2\mathbf{y}$

Solution. $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 8 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ 4 \\ 17 \end{bmatrix}}$

(b) $\mathbf{x} \cdot \mathbf{y}$

Solution. $1(2) + 2(-1) + 3(4) = \boxed{12}$

(c) $A(\mathbf{x} + \mathbf{y})$

Solution.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 1(3) + 1(1) + 2(7) \\ 0(3) + (-1)(1) + 4(7) \end{bmatrix} = \boxed{\begin{bmatrix} 18 \\ 27 \end{bmatrix}}$$

(d) $\mathbf{y} + A\mathbf{x}$ Since

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1(1) + 1(2) + 2(3) \\ 0(1) + (-1)(2) + 4(3) \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$$

$$\mathbf{y} + A\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 9 \\ 10 \end{bmatrix}$$

is undefined.

(e) $B(A\mathbf{x})$

Solution. First, as in part (d),

$$A\mathbf{x} = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$$

so

$$B(A\mathbf{x}) = \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 9 \\ 10 \end{bmatrix} = \begin{bmatrix} 7(9) + 0(10) \\ 4(9) + 1(10) \\ 2(9) + (-4)(10) \end{bmatrix} = \begin{bmatrix} 63 \\ 46 \\ -22 \end{bmatrix}$$

Note: This question did **not** involve matrix multiplication!

10. Circle T or F to mark each of the following true or false.

- | | |
|----------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------|
| (a) If $\mathbf{v} \cdot \mathbf{w} < 0$ then the angle between \mathbf{v} and \mathbf{w} is less than 90° . | T F |
| (b) The dot product of two vectors in \mathbf{R}^3 is a vector in \mathbf{R}^3 . | T F |
| (c) Any three vectors in \mathbf{R}^3 span \mathbf{R}^3 . | T F |
| (d) The columns of a matrix are linearly independent if and only if its rank equals the number of columns. | T F |
| (e) For all matrices A , the column space of A equals the column space of the reduced row echelon form of A . | T F |
| (f) For all matrices A , the null space of A equals the null space of the reduced row echelon form of A . | T F |
| (g) If $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$, then $\dim(V) \leq k$. | T F |
| (h) If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent vectors in V , then $\dim(V) \geq k$. | T F |
| (i) A system of three equations in four unknowns cannot have a unique solution. | T F |
| (j) A system of four equations in three unknowns cannot have more than one solution. | T F |